Special Types of Matrices

Certain matrices, such as the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

have a special "shape", which endows the matrix with helpful properties. The identity matrix is an example of a *diagonal* matrix; we will discuss several types of special matrices in this section, including diagonal matrices, as well as the properties that make them interesting.

Diagonal Matrices

Examine the matrices below:

$$\begin{pmatrix} 3-2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{pmatrix}, \quad \begin{pmatrix} .5 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

You should notice that the three matrices have a common shape–their *only* nonzero entries occur on their diagonals.

Definition. A square matrix is *diagonal* if all of its off-diagonal entries are 0s. A diagonal matrix has form

$$\begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}.$$

We give such matrices a name because they have interesting properties not shared by non-diagonal matrices; we discuss these properties below.

Properties of Diagonal Matrices

We have seen that matrix multiplication is, in general, quite tedious. However, if the two matrices in question are actually *diagonal* matrices, multiplication becomes quite simple, as indicated by the following theorem:

Theorem. The product of a pair of $n \times n$ diagonal matrices

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & 0 & 0 & \dots & 0 \\ 0 & b_{22} & 0 & \dots & 0 \\ 0 & 0 & b_{33} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{nn} \end{pmatrix}$$

is also an $n \times n$ diagonal matrix, and has form

$$AB = \begin{pmatrix} a_{11}b_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22}b_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33}b_{33} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}b_{nn} \end{pmatrix}.$$

Proof. To prove the theorem, we must accomplish two things:

1. Show that each off-diagonal entry is 0.
2. Show that the diagonal entries have form $a_{ii}b_{ii}$.

We know that the i, j entry p_{ij} of the product AB has form

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

We may now divide the problem into two cases:

Case 1: Off-diagonal entries

Case 2: Diagonal entries.

Notice that an off-diagonal entry p_{ij} of a matrix must have $i \neq j$; on the other hand, every diagonal entry has form p_{ii}. We investigate the cases using this observation:

Case 1: Off-diagonal entries: Since p_{ij} is off-diagonal, $i \neq j$. We know that

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj};$$

in addition, A and B are diagonal, so $a_{ik} = 0$ whenever $i \neq k$, and similarly $b_{kj} = 0$ whenever $k \neq j$. Since $i \neq j$, every term in the expansion of

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

has at least one 0 factor. Thus $p_{ij} = 0$ whenever $i \neq j$.

Case 2: Diagonal entries: The diagonal entry p_{ii} has form

$$p_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki};$$

using the reasoning above, the only possible nonzero term in this sum is $a_{ii} b_{ii}$. Thus

$$p_{ii} = a_{ii} b_{ii}.$$

The theorem above has several nice consequences. For starters, finding the inverse of a diagonal matrix is quite simple (unlike finding inverses for most other matrices!). Indeed, a diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; if this is the case, then the inverse of

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix} \text{ is } D^{-1} = \begin{pmatrix} d_1^{-1} & 0 & 0 & \dots & 0 \\ 0 & d_2^{-1} & 0 & \dots & 0 \\ 0 & 0 & d_3^{-1} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n^{-1} \end{pmatrix}.$$

Example. Given

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{4} \end{pmatrix},$$

find A^{-1} and A^3.

1. A^{-1} is simple, and you should check that

$$A^{-1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{5} \end{pmatrix}$$

2. To find A^3, we could start by finding A^2, then calculating $A^2 \cdot A$. However, we know that each product will yield another diagonal matrix, whose diagonal entries are just the products of the corresponding diagonal entries of the factors. So to find A^3, all we really need to do is

cube each of its diagonal entries. We have

$$A^3 = \begin{pmatrix} (\frac{1}{2})^3 & 0 & 0 & 0 & 0 \\ 0 & 7^3 & 0 & 0 & 0 \\ 0 & 0 & 3^3 & 0 & 0 \\ 0 & 0 & 0 & (-1)^3 & 0 \\ 0 & 0 & 0 & 0 & (\frac{5}{4})^3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 343 & 0 & 0 & 0 \\ 0 & 0 & 27 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{125}{64} \end{pmatrix}.$$

Upper and Lower Triangular Matrices

Triangular matrices are our next special type of matrix:

Definition. A square matrix is *upper triangular* if all of its below-diagonal entries are 0s. An upper triangular matrix has form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

A square matrix is *lower triangular* if all of its above-diagonal entries are 0s. A lower triangular matrix has form

$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

The first two matrices below are upper triangular, and the last is lower triangular:

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

In the example above, you should have noticed that the first and third matrices are just transposes. This is an illustration of the first part of the following theorem, which lists some important properties of triangular matrices:

Theorem. (a) The transpose of an upper triangular matrix is lower triangular, and vice versa.

(b) The product of two upper triangular matrices is upper triangular, and the product of two lower triangular matrices is lower triangular.

(c) A triangular matrix is invertible if and only if each of its diagonal entries is nonzero.

(d) The inverse of an invertible upper triangular matrix is lower triangular, and vice versa.

Symmetric Matrices

The matrix

$$A = \begin{pmatrix} 6 & -3i & 2 \\ -3i & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix}$$

has a special property, which you can discover by calculating $A^\top$:

$$A^\top = \begin{pmatrix} 6 & -3i & 2 \\ -3i & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

Notice that $A^\top$ and A are the same matrix, that is

$$A = A^\top.$$

As you may have suspected, we have a name for such special types of matrices:

Definition. A square matrix A is *symmetric* if $A = A^\top$. If A is symmetric, then $a_{ij} = a_{ji}$.

The following theorem lists some interesting properties of symmetric matrices:

Theorem. If A and B are symmetric $n \times n$ matrices, and k is any scalar, then:

(a) $A^\top$ is symmetric.

(b) $A + B$ and $A - B$ are symmetric.

(c) kA is symmetric.

Proof. Let's prove part (b) of the theorem. We'd like to show that, if A and B are symmetric, then so is $A + B$. Of course, if A and B are symmetric, then we know that

$$A = A^\top \text{ and } B = B^\top.$$

Now, to show that $A + B$ is symmetric, we need to be convinced that $(A + B)^\top = (A + B)$. Let's check that this is the case:

$$(A + B)^\top = A^\top + B^\top,$$

since the transpose of a sum is the sum of the transposes. Fortunately, combining this with the observation that $A = A^\top$ and $B = B^\top$ give us

$$\begin{aligned}(A+B)^\top &= A^\top + B^\top \\ &= A + B.\end{aligned}$$

Thus

$$(A+B)^\top = (A+B)$$

as we hoped, which means that $A + B$ is a symmetric matrix.

Hermitian Matrices

Recall that the *conjugate transpose* A^* of matrix A is the matrix

$$A^\top = [\overline{a_{ji}}],$$

whose (i, j)th entry is the conjugate transpose of the (j, i)th entry of A.

Definition. A square matrix A is *hermitian* if $A = A^*$; that is, if $a_{ij} = \overline{a_{ji}}$.

The matrix

$$A = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

is hermitian since

$$A^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} = A.$$

On the other hand, the matrix

$$A = \begin{pmatrix} 6 & -3i & 2 \\ -3i & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix},$$

while symmetric ($A = A^\top$), is *not* hermitian:

$$\begin{aligned}A^* &= \begin{pmatrix} 6 & 3i & 2 \\ 3i & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix} \\ &\neq A.\end{aligned}$$

Even though A from the last example is *symmetric*, it is not *hermitian*; this is due to the signs of the complex entries in A.

However, if matrix A is symmetric and has strictly *real* entries, then A is automatically (trivially) hermitian, as indicated by the following theorem:

Theorem. A real symmetric matrix A is also hermitian.

Proof. Let $A = [a_{ij}]$ be real symmetric. Since A is symmetric, we know that $a_{ij} = a_{ji}$. In addition, each entry of A is real, so $a_{ij} = \overline{a_{ij}}$. Combining these observations, we see that

$$a_{ij} = a_{ji} = \overline{a_{ji}},$$

so that A is hermitian.