Inverses of Matrices

We have seen that many ideas from the world of numbers, such as addition and multiplication, have analogues in matrix theory. The tables below summarizes these ideas:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Outcome</th>
<th>Operation</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real number addition</td>
<td>Real number</td>
<td>Matrix addition</td>
<td>Matrix</td>
</tr>
<tr>
<td>Real number multiplication</td>
<td>Real number</td>
<td>Matrix multiplication</td>
<td>Matrix</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Object</th>
<th>Importance</th>
<th>Object</th>
<th>Importance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (as a number)</td>
<td>Additive identity</td>
<td>0 (matrix)</td>
<td>Additive identity</td>
</tr>
<tr>
<td>1</td>
<td>Multiplicative identity</td>
<td>I</td>
<td>Multiplicative identity</td>
</tr>
</tbody>
</table>

It turns out that there are more analogues between the real numbers and matrices. For example, we know that most real numbers have a multiplicative inverse: for example, the multiplicative inverse of 5 is $1/5$ since

$$5 \times \frac{1}{5} = 1.$$  

The numbers 5 and $1/5$ are multiplicative inverses since their product is the multiplicative identity 1. (Question: which number(s) have no multiplicative inverse?)

It turns out that there is an analogous idea for square matrices, as indicated by the following definition:

**Definition.** An $n \times n$ matrix $A$ has a multiplicative inverse $B$ if

$$AB = BA = I_n.$$  

For example, the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

has inverse

$$B = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -3 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$  

You should verify via multiplication that

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$
Remark. A few quick notes here:

- If $B$ is the multiplicative inverse of the $n \times n$ matrix $A$, so that $AB = I_n$, then necessarily $B$ must be $n \times n$.
- If $AB = I$, then it automatically follows that $BA = I$.
- Inverses are unique. That is, if $B$ and $C$ are both inverses of $A$, then $B = C$. We prove this fact below.

Theorem. Suppose that $A$ is an $n \times n$ matrix, and that $B$ and $C$ are both multiplicative inverses of $A$. Then $B = C$.

Proof. Since $B$ is an inverse of $A$, we know that

$$AB = BA = I.$$ 

Similarly,

$$AC = CA = I.$$ 

Clearly

$$AB = AC;$$

multiplying both sides of this equation on the left by $B$, we see that

$$B(AB) = B(AC)$$
$$= (BA)B = (BA)C$$
$$IB = IC$$
$$B = C.$$ 

Remark. The theorem above is actually a cancellation law: if

$$AB = AC$$

and $A$ is invertible, then

$$B = C.$$ 

Note that this cancellation law is only guaranteed if $A$ has an inverse. We saw an example earlier where the cancellation law did not work: with

$$A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix},$$
it is easy to see that

\[ AB = AC, \]

but of course \( B \neq C \). The reason that cancellation fails is that \( A \) is not invertible: the first row of \( A \), which consists entirely of 0s, will force the first row of any product \( AB \) to consist entirely of 0s as well; so \( AB \) can never equal \( I \).

A matrix \( A \) that has an inverse is called \textit{invertible} or \textit{nonsingular}, and we refer to its inverse using the notation \( A^{-1} \). Using the example above, we write

\[
A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -3 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.
\]

(Note: the notation \( A^{-1} \) does not refer to division in any way. Indeed, we do not even have a definition for matrix division, nor will we see one!)

Just as the number 0 has no multiplicative inverse, there are many matrices which do not have multiplicative inverses; for example, we saw above that

\[
A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}
\]

has no inverse. A matrix which does not have inverse is called \textit{singular}.

One important question that we will need to answer is this: how can we determine whether a specific matrix \( A \) is singular or nonsingular? The question has a surprising answer, which we will study in this and several later sections.

---

**Finding Inverses of 2 × 2 Matrices**

As indicated earlier, we would like to have a reliable method for

1. determining whether or not a specific matrix has an inverse, and
2. finding inverses when they exist.

In general, it is quite simple to tell if a matrix is invertible or not, but much more difficult to find that inverse. However, for the special case of 2 × 2 matrices, calculating inverses is actually quite easy. Thus we will focus on the 2 × 2 case for now, and leave larger matrices for a later discussion.

**Theorem.** (a) The matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
has an inverse if and only if \( ad - bc \neq 0 \).

(b) If \( ad - bc \neq 0 \), then the inverse of \( A \) is the matrix

\[
A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

Proof. Let’s prove part (b) of the theorem: we would like to show that, if \( ad - bc \neq 0 \), then

\[
A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

To do so, we need to calculate \( AA^{-1} \); if the theorem is true, then

\[
AA^{-1} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let’s check the theorem by multiplying:

\[
AA^{-1} = \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

\[
= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix}
\]

\[
= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}
\]

\[
= \frac{1}{ad - bc} (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
= I_2.
\]

Thus our claim is correct—since \( AA^{-1} = I_2 \), the theorem does indeed give the correct formula for \( A^{-1} \).

Example. Determine if the following matrices are invertible, and find their inverses if possible:

1. \( A = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix} \)

2. \( B = \begin{pmatrix} 6 & 4 \\ -4 & -\frac{2}{3} \end{pmatrix} \)
1. To determine if

\[ A = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix} \]

is invertible, we need to calculate the number \( ad - bc \): with

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix},
\]
we have \( ad - bc = 12 - 20 = -8 \).

Since \( ad - bc = -8 \neq 0 \), this matrix is invertible; the formula from the theorem tells us that

\[ A^{-1} = \frac{-1}{8} \begin{pmatrix} -3 & -10 \\ -2 & -4 \end{pmatrix}. \]

2. Unlike the previous matrix, it is clear that

\[ B = \begin{pmatrix} 6 & 4 \\ -4 & -\frac{8}{3} \end{pmatrix} \]

is not invertible: in this case,

\[ ad - bc = -16 + 16 = 0. \]

Properties of Inverses

We have yet to answer several important questions about the way that matrix inverses work. We might wish to know

- the inverse of a product \( AB \);
- the inverse of a power \( A^r \);
- or the inverse of a transpose \( A^\top \).

We will answer these and several more related questions about inverses in the remainder of this section.

Inverses of Products

If we know that \( A \) and \( B \) are invertible \( n \times n \) matrices with inverses \( A^{-1} \) and \( B^{-1} \), respectively, it seems reasonable to guess that the product \( AB \) also has an inverse. Let’s try to find a formula for the inverse \((AB)^{-1}\) of \( AB \): we want a matrix \( X \) so that

\[ (AB)X = I, \text{ or } A(BX) = I. \]

Since we are trying to determine a value for \( X \), let’s just leave it as a blank:

\[ A(B\underline{\text{_____}}) = I. \] (1)
It seems reasonable to guess that $A^{-1}$ and $B^{-1}$ should play a roll in the desired formula. In fact, we can quickly eliminate the factor of $B$ in (1) using $B^{-1}$:

$$A(BB^{-1}) = A(I) = A.$$ 

Of course, we still want

$$A(BB^{-1}) = A = I.$$ 

We’ve certainly moved a bit closer to our goal—all we need to do now is fill in the blank in a way that eliminates the factor of $A$. Of course, we can do this easily: since

$$AA^{-1} = I,$$

we know that $A^{-1}$ is the missing factor.

Let’s look at what we’ve done: we filled in the blank in

$$A(B__) = I$$

with $B^{-1}A^{-1}$. Indeed,

$$AB(B^{-1}A^{-1}) = A(B(B^{-1}A^{-1}))$$
$$= A((BB^{-1})A^{-1})$$
$$= A(IA^{-1})$$
$$= AA^{-1}$$
$$= I,$$

so it is clear that

**Theorem.** If $A$ and $B$ are both invertible matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$ 

**Remark.** We can actually generalize the theorem a bit: if each of $A_1$, $A_2$, $\ldots$, $A_n$ is invertible, then so is the product

$$A_1A_2\ldots A_n,$$

and

$$(A_1A_2\ldots A_n)^{-1} = A_n^{-1}\ldots A_2^{-1}A_1^{-1}.$$ 

In other words, the inverse of a product is the product of the inverses in reverse order.
Inverses of Matrices

We can define integer powers of a square matrix in a natural way, analogous to the way that powers of real numbers are defined:

**Definition.** If \( r \) is an integer, \( r > 0 \), and \( A \) is an \( n \times n \) matrix, then

\[
A^r = A \cdot A \cdot \ldots \cdot A.
\]

We define \( A^0 \) to be

\[
A^0 = I_n.
\]

It is quite easy to see that the usual rules for combining powers of real numbers also work for combining powers of matrices:

**Theorem.** If \( r \) and \( s \) are nonnegative real numbers, and \( A \) is square, then

- \( A^r A^s = A^{r+s} \)
- \( (A^r)^s = A^{rs} \).

Of course, if \( A \) is invertible, with inverse \( A^{-1} \), then we might wish to know the inverse of some power of \( A \). As an example, let’s find a formula for the inverse of \( A^3 \): we want

\[
A^3 = I;
\]

of course, if we replace the blank with three copies of \( A^{-1} \), we should get the desired result. Let’s check:

\[
A^3 (A^{-1})^3 = (A \cdot A \cdot A)(A^{-1} \cdot A^{-1} \cdot A^{-1})
\]

\[
= (A \cdot A) \cdot (A^{-1} \cdot A^{-1}) \cdot (A^{-1} \cdot A^{-1})
\]

\[
= (A \cdot A) \cdot I \cdot (A^{-1} \cdot A^{-1})
\]

\[
= (A \cdot A) \cdot (A^{-1} \cdot A^{-1})
\]

\[
= A \cdot (A^{-1} \cdot A^{-1})
\]

\[
= A \cdot I \cdot A^{-1}
\]

\[
= A \cdot A^{-1}
\]

\[
= I.
\]

Thus we have verified that

\[
(A^3)^{-1} = (A^{-1})^3;
\]

in general,

**Theorem.** If \( A \) is an invertible \( n \times n \) matrix, \( r \) is a nonnegative integer, and \( k \) is a nonzero scalar, then
Inverses of Matrices

- \((A^r)^{-1} = (A^{-1})^r\),
- \((A^{-1})^{-1} = A\), and
- \((kA)^{-1} = k^{-1}A^{-1}\).

Instead of using the clumsy notation \((A^{-1})^r\), we use \(A^{-r}\).

**Key Point.** The notation \(A^{-r}\) indicates the matrix that is the inverse of \(A^r\).

Inverses of Transposes

Recall that the transpose of a matrix \(A\) is the matrix \(A^\top\) that we build by switching the rows and columns of \(A\). For example, if

\[
A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}, \text{ then } A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 5 \end{pmatrix}.
\]

If \(A\) is a square invertible matrix, then we should be able to find a formula for the inverse \((A^\top)^{-1}\) of \(A^\top\); the following theorem provides us with that formula:

**Theorem.** If the \(n \times n\) matrix \(A\) is invertible, then so is \(A^\top\), and

\[
(A^\top)^{-1} = (A^{-1})^\top.
\]

In other words, to find the inverse of the transpose of \(A\), just calculate the inverse of \(A\) and take its transpose!

**Proof.** Since \(A\) is invertible, \(A^{-1}\) exists, and \(AA^{-1} = I\). Taking transposes of both sides, we see that

\[
(AA^{-1})^\top = I^\top
\]

\[
(AA^{-1})^\top = I
\]

\[
(A^{-1})^\top A^\top = I;
\]

the last line guarantees that \((A^{-1})^\top\) is the unique inverse of \(A^\top\), and we have

\[
(A^{-1})^\top = (A^\top)^{-1}.
\]