# Inverses of Matrices

We have seen that many ideas from the world of numbers, such as addition and multiplication, have analogues in matrix theory. The tables below summarizes these ideas:

| Real numbers | | Matrices | |
|---|---|---|---|
| Operation | Outcome | Operation | Outcome |
| Real number addition | Real number | Matrix addition | Matrix |
| Real number multiplication | Real number | Matrix multiplication | Matrix |

| Real numbers | | Matrices | |
|---|---|---|---|
| Object | Importance | Object | Importance |
| 0 (as a number) | Additive identity | **0** (matrix) | Additive identity |
| 1 | Multiplicative identity | I | Multiplicative identity |

It turns out that there are more analogues between the real numbers and matrices. For example, we know that *most* real numbers have a multiplicative inverse: for example, the multiplicative inverse of 5 is 1/5 since

$$5 \times \frac{1}{5} = 1.$$

The numbers 5 and 1/5 are multiplicative inverses since their product is the multiplicative identity 1. (Question: which number(s) have no multiplicative inverse?)

It turns out that there is an analogous idea for square matrices, as indicated by the following definition:

**Definition.** An $n \times n$ matrix $A$ has a *multiplicative inverse* $B$ if

$$AB = BA = I_n.$$

For example, the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

has inverse

$$B = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -3 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

You should verify via multiplication that

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

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**Remark.** A few quick notes here:

- If $B$ is the multiplicative inverse of the $n \times n$ matrix $A$, so that $AB = I_n$, then necessarily $B$ must be $n \times n$.
- If $AB = I$, then it automatically follows that $BA = I$.
- Inverses are unique. That is, if $B$ and $C$ are both inverses of $A$, then $B = C$. We prove this fact below.

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**Theorem.** Suppose that $A$ is an $n \times n$ matrix, and that $B$ and $C$ are both multiplicative inverses of $A$. Then $B = C$.

**Proof.** Since $B$ is an inverse of $A$, we know that

$$AB = BA = I.$$

Similarly,

$$AC = CA = I.$$

Clearly

$$AB = AC;$$

multiplying both sides of this equation on the left by $B$, we see that

$$\begin{aligned} B(AB) &= B(AC) \\ (BA)B &= (BA)C \\ IB &= IC \\ B &= C. \end{aligned}$$

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**Remark.** The theorem above is actually a cancellation law: if

$$AB = AC$$

and $A$ is invertible, then

$$B = C.$$

Note that this cancellation law is *only* guaranteed if $A$ has an inverse. We saw an example earlier where the cancellation law *did not* work: with

$$A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix},\ \text{and } C = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix},$$

it is easy to see that

$$AB = AC,$$

but of course $B \neq C$. The reason that cancellation fails is that $A$ *is not* invertible: the first row of $A$, which consists entirely of 0s, will force the first row of any product $AB$ to consist entirely of 0s as well; so $AB$ can never equal $I$.

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A matrix $A$ that has an inverse is called *invertible* or *nonsingular*, and we refer to its inverse using the notation $A^{-1}$. Using the example above, we write

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -3 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

(Note: the notation $A^{-1}$ does *not* refer to division in any way. Indeed, we do not even have a definition for matrix division, nor will we see one!)

Just as the *number* 0 has no multiplicative inverse, there are many matrices which do not have multiplicative inverses; for example, we saw above that

$$A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}$$

has no inverse. A matrix which does not have inverse is called *singular*.

One important question that we will need to answer is this: how can we determine whether a specific matrix $A$ is singular or nonsingular? The question has a surprising answer, which we will study in this and several later sections.

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## Finding Inverses of $2 \times 2$ Matrices

As indicated earlier, we would like to have a reliable method for

1. determining whether or not a specific matrix has an inverse, and
2. *finding* inverses when they exist.

In general, it is quite simple to tell if a matrix is invertible or not, but much more difficult to *find* that inverse. However, for the special case of $2 \times 2$ matrices, calculating inverses is actually quite easy. Thus we will focus on the $2 \times 2$ case for now, and leave larger matrices for a later discussion.

**Theorem.** (a) The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse *if and only if*

$$ad - bc \neq 0.$$

(b) If $ad - bc \neq 0$, then the inverse of $A$ is the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Proof.** Let's prove part (b) of the theorem: we would like to show that, if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To do so, we need to calculate $AA^{-1}$; if the theorem is true, then

$$AA^{-1} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let's check the theorem by multiplying:

$$\begin{aligned}
AA^{-1} &= \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix} \\
&= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \frac{1}{ad - bc}(ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I_2.
\end{aligned}$$

Thus our claim is correct–since $AA^{-1} = I_2$, the theorem does indeed give the correct formula for $A^{-1}$.

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**Example.** Determine if the following matrices are invertible, and find their inverses if possible:

1. $A = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix}$

2. $B = \begin{pmatrix} 6 & 4 \\ -4 & -\frac{8}{3} \end{pmatrix}$

1. To determine if
$$A = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix}$$
is invertible, we need to calculate the number $ad - bc$: with
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix}, \text{ we have } ad - bc = 12 - 20 = -8.$$
Since $ad - bc = -8 \neq 0$, this matrix is invertible; the formula from the theorem tells us that
$$A^{-1} = -\tfrac{1}{8} \begin{pmatrix} -3 & -10 \\ -2 & -4 \end{pmatrix}.$$

2. Unlike the previous matrix, it is clear that
$$B = \begin{pmatrix} 6 & 4 \\ -4 & -\frac{8}{3} \end{pmatrix}$$
is not invertible: in this case,
$$ad - bc = -16 + 16 = 0.$$

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## Properties of Inverses

We have yet to answer several important questions about the way that matrix inverses work. We might wish to know

- the inverse of a product $AB$;
- the inverse of a power $A^r$;
- or the inverse of a transpose $A^\top$.

We will answer these and several more related questions about inverses in the remainder of this section.

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### Inverses of Products

If we know that $A$ and $B$ are invertible $n \times n$ matrices with inverses $A^{-1}$ and $B^{-1}$, respectively, it seems reasonable to guess that the product $AB$ also has an inverse. Let's try to find a formula for the inverse $(AB)^{-1}$ of $AB$: we want a matrix $X$ so that
$$(AB)X = I, \text{ or } A(BX) = I.$$
Since we are trying to determine a value for $X$, let's just leave it as a blank:
$$A(B\underline{\quad\quad}) = I. \tag{1}$$

It seems reasonable to guess that $A^{-1}$ and $B^{-1}$ should play a roll in the desired formula. In fact, we can quickly eliminate the factor of $B$ in (1) using $B^{-1}$:

$$A(BB^{-1}\_\_) = A(I\_\_) = A\_\_.$$

Of course, we still want

$$A(BB^{-1}\_\_) = A\_\_ = I.$$

We've certainly moved a bit closer to our goal–all we need to do now is fill in the blank in a way that eliminates the factor of $A$. Of course, we can do this easily: since

$$AA^{-1} = I,$$

we know that $A^{-1}$ is the missing factor.

Let's look at what we've done: we filled in the blank in

$$A(B\_\_\_\_) = I$$

with $B^{-1}A^{-1}$. Indeed,

$$\begin{aligned} AB(B^{-1}A^{-1}) &= A(B(B^{-1}A^{-1})) \\ &= A((BB^{-1})A^{-1}) \\ &= A(IA^{-1}) \\ &= AA^{-1} \\ &= I, \end{aligned}$$

so it is clear that

**Theorem.** If $A$ and $B$ are both invertible matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**Remark.** We can actually generalize the theorem a bit: if each of $A_1$, $A_2$, $\ldots$, $A_n$ is invertible, then so is the product

$$A_1A_2\ldots A_n,$$

and

$$(A_1A_2\ldots A_n)^{-1} = A_n^{-1}\ldots A_2^{-1}A_1^{-1}.$$

In other words, the inverse of a product is the product of the inverses in reverse order.

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**Inverses of Powers of Matrices**

We can define integer powers of a square matrix in a natural way, analogous to the way that powers of real numbers are defined:

**Definition.** If $r$ is an integer, $r > 0$, and $A$ is an $n \times n$ matrix, then

$$A^r = \underbrace{A \cdot A \cdot \ldots \cdot A}_{r \text{ factors}}.$$

We define $A^0$ to be

$$A^0 = I_n.$$

It is quite easy to see that the usual rules for combining powers of real numbers also work for combining powers of matrices:

**Theorem.** If $r$ and $s$ are nonnegative real numbers, and $A$ is square, then

- $A^r A^s = A^{r+s}$
- $(A^r)^s = A^{rs}$.

Of course, if $A$ is invertible, with inverse $A^{-1}$, then we might wish to know the inverse of some power of $A$. As an example, let's find a formula for the inverse of $A^3$: we want

$$A^3\underline{\hspace{2cm}} = I;$$

of course, if we replace the blank with three copies of $A^{-1}$, we should get the desired result. Let's check:

$$\begin{aligned}
A^3(A^{-1})^3 &= (A \cdot A \cdot A)(A^{-1} \cdot A^{-1} \cdot A^{-1}) \\
&= (A \cdot A) \cdot (A \cdot A^{-1}) \cdot (A^{-1} \cdot A^{-1}) \\
&= (A \cdot A) \cdot I \cdot (A^{-1} \cdot A^{-1}) \\
&= (A \cdot A) \cdot (A^{-1} \cdot A^{-1}) \\
&= A \cdot (A \cdot A^{-1}) \cdot A^{-1} \\
&= A \cdot I \cdot A^{-1} \\
&= A \cdot A^{-1} \\
&= I.
\end{aligned}$$

Thus we have verified that

$$(A^3)^{-1} = (A^{-1})^3;$$

in general,

**Theorem.** If $A$ is an invertible $n \times n$ matrix, $r$ is a nonnegative integer, and $k$ is a nonzero scalar, then

- $(A^r)^{-1} = (A^{-1})^r$,
- $(A^{-1})^{-1} = A$, and
- $(kA)^{-1} = k^{-1}A^{-1}$.

Instead of using the clumsy notation $(A^{-1})^r$, we use $A^{-r}$.

**Key Point.** The notation $A^{-r}$ indicates the matrix that is the inverse of $A^r$.

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**Inverses of Transposes**

Recall that the transpose of a matrix $A$ is the matrix $A^\top$ that we build by switching the rows and columns of $A$. For example, if

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}, \text{ then } A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 5 \end{pmatrix}.$$

If $A$ is a square invertible matrix, then we should be able to find a formula for the inverse $(A^\top)^{-1}$ of $A^\top$; the following theorem provides us with that formula:

**Theorem.** If the $n \times n$ matrix $A$ is invertible, then so is $A^\top$, and

$$(A^\top)^{-1} = (A^{-1})^\top.$$

In other words, to find the inverse of the transpose of $A$, just calculate the inverse of $A$ and take *its* transpose!

**Proof.** Since $A$ is invertible, $A^{-1}$ exists, and $AA^{-1} = I$. Taking transposes of both sides, we see that

$$\begin{aligned} (AA^{-1})^\top &= I^\top \\ (AA^{-1})^\top &= I \\ (A^{-1})^\top A^\top &= I; \end{aligned}$$

the last line guarantees that $(A^{-1})^\top$ is the unique inverse of $A^\top$, and we have

$$(A^{-1})^\top = (A^\top)^{-1}.$$