Matrix Operations

A matrix is a new mathematical object, in the same way that numbers are mathematical objects. Just as we can add, subtract, and multiply numbers, we can add, subtract, and multiply matrices.

However, since matrices *are* new mathematical objects, we will need special definitions for the operations of matrix addition and matrix multiplication. There is also an extra operation on matrices known as scalar multiplication. We introduce each of these operations below, and discuss their properties in the next section.

Scalar Multiplication

Scalar multiplication is an operation which allows us to multiply a scalar (number) by a matrix:

Definition. If c is a number (either real or complex) and $A = [a_{ij}]$ is a matrix, then the scalar product cA is the matrix whose i, jth entry is $c \cdot a_{ij}$.

Example. Given

$$A = \begin{pmatrix} 3 & -i & 4 \\ 5 & 1 & 3 \\ 2+i & 2 & 0 \end{pmatrix},$$

find the scalar product $-2A$.

To find $-2A$, we simply multiply each entry of A by -2, yielding the matrix

$$-2A = \begin{pmatrix} -6 & 2i & -8 \\ -10 & -2 & -6 \\ -4-2i & -4 & 0 \end{pmatrix}.$$

Matrix Addition and Subtraction

The definition for matrix addition is exactly what one would expect:

Definition. Given $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, their sum $A + B$ is the matrix

$$A + B = [a_{ij} + b_{ij}]$$

whose i, j entry is just the sum of the i, j entries of A and B.

It is important to notice that we can *only* add matrices A and B if they have the same size. In this respect, matrix addition differs from number addition. While we can add any pair of numbers we wish, we *can only* add a pair of matrices if they are of comparable sizes.

Example. Given

$$A = \begin{pmatrix} 3+i & -1 & 4 \\ 5 & 1 & 3 \\ 2-2i & 2 & 0 \\ 6 & -3 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -2 & 5 \\ -2 & -i & 1 \\ 2i & 3 & 3 \\ 1 & 0 & 1 \end{pmatrix},$$

find $A+B$.

To find $A+B$, we simply need to add corresponding entries of A and B. For example, the $(1,1)$ entry of $A+B$ should be $(3+i)+2 = 5+i$, and the $(1,2)$ entry should be $-1-2=-3$.

Thus $A+B$ is given by

$$A+B = \begin{pmatrix} 3+i & -1 & 4 \\ 5 & 1 & 3 \\ 2-2i & 2 & 0 \\ 6 & -3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 5 \\ -2 & -i & 1 \\ 2i & 3 & 3 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5+i & -3 & 9 \\ 3 & 1-i & 4 \\ 2 & 5 & 3 \\ 7 & -3 & 1 \end{pmatrix}.$$

Matrix subtraction is defined using scalar multiplication and matrix addition: given $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, their difference $A - B$ is the matrix

$$A + (-B) = [a_{ij} - b_{ij}]$$

whose i,j entry is just the difference of the i,j entries of A and B.

Matrix Multiplication

The definition for matrix multiplication that you might expect–simply multiply corresponding entries!–is actually not the standard way to multiply a pair of matrices. The standard definition, which we discuss momentarily, seems rather arbitrary, but turns out to have important properties that lend themselves well to understanding vector spaces. We begin to build the definition for matrix multiplication by discussing how to multiply row and column matrices.

The Scalar Product of a Row and a Column Matrix

Recall that a row matrix is one consisting of a single row, and has form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix};$$

similarly, a column matrix has form

$$B = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}.$$

If $A = [a_{1j}]$ is a row matrix (with size, say, $1 \times n$), and $B = [b_{i1}]$ is a column matrix (with size $n \times 1$), then the *scalar product* of A and B is the *number*

$$A \cdot B = \sum_{k=1}^{n} a_{1k} b_{k1} = a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1}.$$

Key Point. If this definition reminds you of the dot product in $\mathbb{R}^n$, you're on the right track–the definitions are virtually the same.

Example. Find the scalar product of

$$\begin{pmatrix} 3i & -1 & 2 & i \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 4 \\ 0 \\ -i \end{pmatrix}.$$

The scalar product is

$$\begin{aligned} 3i \cdot 2 + (-1) \cdot 4 + 2 \cdot 0 + i \cdot (-i) &= 6i - 4 + 0 + 1 \\ &= -3 + 6i. \end{aligned}$$

Product of Two Matrices

The definition of matrix multiplication, which involves the scalar product, will allow us to multiply any pair A and B of matrices whose sizes are amenable:

Definition. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then their product AB is the $m \times n$ matrix whose (i, j)th entry is the scalar product of the ith row of A and the jth column of B. In other words, if AB has entries p_{ij}, then

$$p_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

If the sizes of A and B are *not* amenable, then the product AB is not defined and cannot be computed.

Example. Find the product AB of matrices

$$A = \begin{pmatrix} 1 & 4i \\ 3 & 2 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & -2 & 1 \\ 0 & -4 & 3i \end{pmatrix}.$$

Notice that, since A is a 3×2 matrix and B is a 2×3 matrix, we *can* indeed find the product AB, which will be a 3×3 matrix.

We populate the entries of AB by finding the scalar product of rows of A with columns of B; we will refer to the entries of AB using c_{ij}s. For example, the $1,1$ entry c_{11} of AB is the scalar product of the first row of A with the first column of B:

$$c_{11} = 1 \cdot 5 + 4i \cdot 0 = 5;$$

we fill in the entry of AB accordingly:

$$AB = \begin{pmatrix} 5 & & \\ & & \\ & & \end{pmatrix}.$$

Let's calculate c_{12}, the scalar product of the first row of A with the second column of B:

$$c_{12} = 1 \cdot -2 + 4i \cdot -4 = -2 - 16i;$$

we have

$$AB = \begin{pmatrix} 5 & -2-16i & \\ & & \\ & & \end{pmatrix}.$$

The next entry c_{13} is the scalar product of the first row of A and the third column of B,

$$c_{13} = 1 \cdot 1 + 4i \cdot 3i = 1 - 12 = -11.$$

We have

$$AB = \begin{pmatrix} 5 & -2-16i & -11 \\ & & \\ & & \end{pmatrix}.$$

Continuing the process, we see that

$$c_{21} = 3 \cdot 5 + 2 \cdot 0 = 15,$$

$$c_{22} = 3 \cdot -2 + 2 \cdot -4 = -14$$

$$c_{23} = 3 \cdot 1 + 2 \cdot 3i = 3 + 6i,$$

$$c_{31} = -1 \cdot 5 + 0 \cdot 0 = -5,$$

$$c_{32} = -1 \cdot -2 + 0 \cdot -4 = 2, \text{ and}$$

$$c_{33} = -1 \cdot 1 + 0 \cdot 3i = -1.$$

Thus the matrix AB is given by

$$AB = \begin{pmatrix} 5 & -2-16i & -11 \\ 15 & -14 & 3+6i \\ -5 & 2 & -1 \end{pmatrix}.$$

If A is an $1 \times n$ row matrix and B is a $n \times 1$ column matrix, then both their scalar product $A \cdot B$ and matrix product AB are defined; it is important to distinguish between the two operations. The *scalar product* $A \cdot B$ is a number, but their *matrix product* AB is the 1×1 matrix whose only entry is the scalar product (number) $A \cdot B$.

For example, with

$$A = \begin{pmatrix} 3 & -1 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 \\ 4 \\ 0 \\ -2 \end{pmatrix},$$

the scalar product $A \cdot B$ is the number

$$A \cdot B = 3 \cdot 2 + (-1) \cdot 4 + 2 \cdot 0 + 1 \cdot (-2) = 0,$$

but the matrix product AB is the matrix

$$AB = \begin{pmatrix} 0 \end{pmatrix}.$$

Key Point. The sizes of the matrices A and B give us a good bit of information about what to expect when we attempt to multiply A by B.

- The product AB is *defined* if the inner indices match up.
  - If A is 3×5 and B is 5×3, then the product AB is defined; notice that the inner indices, highlighted in blue below, match up:

    $$3 \times 5 \qquad\qquad 5 \times 2$$

    product defined

  - On the other hand, the product BA is *not* defined, as the inner indices no longer match:

    $$5 \times 2 \qquad\qquad 3 \times 5$$

    product not defined

- If the product AB is defined, then the *size* of the new matrix AB is determined by the outer indices.

- Using the same matrices A and B above, the product AB must be 3×2, as indicated by the outer indices highlighted in red below:

$$3 \times 5 \qquad\qquad 5 \times 2$$
$$AB \text{ is } 3 \times 2$$

- If X is 5×6 and Y is 6×1, the product XY is defined, and has size 5×1:

$$5 \times 6 \qquad\qquad 6 \times 1$$
$$AB \text{ is } 5 \times 1$$

Key Point. It is important to note at this point that, in general, matrix multiplication is not commutative–in other words,

$$AB \neq BA.$$

We can use the earlier example with

$$A = \begin{pmatrix} 1 & 4i \\ 3 & 2 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & -2 & 1 \\ 0 & -4 & 3i \end{pmatrix}$$

to see that this is true without any calculation. Since A is a 3×2 matrix and B is a 2×3 matrix, AB is a 3×3 matrix, but BA is only 2×2. You should check your understanding of matrix multiplication by verifying that

$$BA = \begin{pmatrix} -2 & -4 + 20i \\ -12 - 3i & -8 \end{pmatrix}.$$

On a similar note, it is completely possible that the product of a pair of matrices A and B could yield the 0 matrix, even though neither of A nor B is 0. For example, you should verify via matrix multiplication that

$$\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This phenomenon is a stark departure from the real number case–we know that *any* product ab of real numbers that yields a 0 *must* include a factor of 0, i.e. either $a = 0$ or $b = 0$.