Matrices and Linear Systems, and an Inversion Algorithm

There are some particularly deep interconnections between linear systems and their corresponding matrices. To understand the connections in more detail, let's recall how to build the matrix of a linear system:

A linear system of m equations in n unknowns, such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

has $m \times n$ augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\ \vdots & & \ddots & & | & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \end{pmatrix}.$$

We will begin by verifying that this encoding makes sense. Starting with the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

we're going to create 3 related matrices:

- the $m \times n$ *coefficient matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix};$$

- the $n \times 1$ matrix of unknowns

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix};$$

- and the $n \times 1$ constant matrix

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Let's calculate the product $A\mathbf{x}$:

$$\begin{aligned}
A\mathbf{x} &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
&= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix} \\
&= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \\
&= \mathbf{b}.
\end{aligned}$$

In other words, we can rewrite the system in (1) as the *matrix equation*

$$A\mathbf{x} = \mathbf{b}.$$

Notice that the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & & \ddots & & | & \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{pmatrix}$$

is calculated by suppressing $\mathbf{x}$ and adjoining A and $\mathbf{b}$,

$$\begin{pmatrix} A & | & \mathbf{b} \end{pmatrix}.$$

The solution technique we learned in the previous section effectively determines the values of the unknowns x_1, x_2, etc., using matrix operations on the augmented matrix.

The following theorem indicates a few of the many ways in which the coefficient matrix for a linear system of n equations in n unknowns provides us with information about the system itself:

Theorem. Let A be an $n \times n$ matrix. Then the following are equivalent:

- A is invertible.
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

- The reduced row echelon form of A is I_n.
- $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
- $\det A \neq 0$.

Recall that the phrase "the following are equivalent" means that either *all the statements are true about a particular matrix*, or *all of them are false.*

For example, if A is the coefficient matrix for a linear system, and we calculate that $\det A = 0$, then we automatically know that:

- A is not invertible.
- $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
- The reduced row echelon form of A has at least 1 row of 0s.
- $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $n \times 1$ matrix $\mathbf{b}$.

Thus the theorem gives us a great deal of information about a system and its coefficient matrix if we happen to know just one bit of information about the matrix itself.

We do not yet have the tools to prove the entire theorem, but certain parts are easy to ascertain. For example, suppose that we know that the reduced row echelon form of A is I_n, that is, there is some sequence of row operations that reduces the $n \times n$ matrix A to

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Clearly the system $I\mathbf{x} = \mathbf{b}$,

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

has *precisely one solution* $\mathbf{x}$ for each $n \times 1$ matrix $\mathbf{b}$: just choose $\mathbf{x} = \mathbf{b}$. The matrix equations $A\mathbf{x} = \mathbf{b}$ and $I\mathbf{x} = \mathbf{b}$ share solution sets, so the system $A\mathbf{x} = \mathbf{b}$ also has precisely one solution $\mathbf{x}$ for each $n \times 1$ matrix $\mathbf{b}$.

Key Point. We noted earlier in the course that the only *number* that does not have an inverse is 0. Now we have an matrix analogue for this fact: the only *square matrices* without inverses are those with determinant 0.

An Inversion Algorithm

Now that we know an easy way to determine whether or not an $n \times n$ matrix A is invertible (just check $\det A$!) we are ready to see an algorithm that will produce the inverse of A, when it exists.

Theorem. Let A be an $n \times n$ matrix. The following algorithm will produce the matrix A^{-1}:

1. Create the matrix $\begin{pmatrix} A & \mid & I \end{pmatrix}$ by adjoining the $n \times n$ identity matrix to A.
2. Apply elementary row operations to the entire matrix, reducing A to the identity.
3. The matrix now has form $\begin{pmatrix} I & \mid & A^{-1} \end{pmatrix}$, where A^{-1} is obtained by applying the same sequence of elementary row operations from step 2 to I.

The proof of this theorem is beyond the scope of this course.

Example. Find the inverse of

$$A = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

We begin by augmenting A with I_4; as we apply elementary row operations to A, we will apply them to I as well:

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Let's start by creating a leading 1 in the first row:

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Next, we should create 0s below this leading 1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 1-1 & 3 & 1 & 0+1 & 0-\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1-1 & 2 & 0 & 0+1 & 0-\frac{1}{2} & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right).$$

Next, we want a leading 1 in row 2:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right).$$

Let's create 0 entries below the leading 1 from row 2:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2-2 & 0-\frac{2}{3} & 1-\frac{2}{3} & -\frac{1}{2}+\frac{1}{3} & 0-\frac{2}{3} & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right).$$

Next we need a leading 1 in row 3:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right),$$

followed by 0 entries below it:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3}+\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0-\frac{2}{9} & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & -\frac{2}{9} & 1 \end{array}\right)$$

and a leading 1 in the last row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & -\frac{2}{9} & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right).$$

Finally, we need to create 0 entries above all of the leading 1s; starting with the last row, we have

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3}-\frac{1}{3} & -\frac{1}{6}+\frac{1}{6} & \frac{1}{3}+\frac{2}{3} & 0+\frac{2}{9} & 0-1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1+1 & \frac{1}{2}-\frac{1}{2} & 0-2 & 0-\frac{2}{3} & 0+3 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right).$$

Finally, we need to create 0 entries above the leading 1 from row 3:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3}-\frac{1}{3} & 0 & 0 & 1 & \frac{2}{9}+\frac{1}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & \frac{1}{3} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right).$$

Notice that we have now reduced A to the identity; the remaining submatrix on the right hand side of the augmented matrix is A^{-1}. We conclude (and you should check) that

$$A^{-1} = \begin{pmatrix} 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & -1 \\ 0 & 0 & -\frac{1}{3} & 0 \\ -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{pmatrix}.$$