Trigonometry Review

Given an arc of length $s$ on a circle of radius $r$, the radian measure of the central angle subtended by the arc is given by $\theta = \frac{s}{r}$:

To convert from radians (rad) to degrees ($^\circ$) and vice versa, use the following conversions:

$$1 \text{ rad} = \frac{180^\circ}{\pi}, \quad 1^\circ = \frac{\pi}{180} \text{ rad}.$$  

In particular, $180^\circ = \pi$ rad.

Given a right triangle (a triangle one of whose angles is $\frac{\pi}{2}$ rad), choose one of the acute angles ($< \frac{\pi}{2}$ rad), and call it $\theta$. We label the sides relative to $\theta$ as follows:

Let $o$, $a$, and $h$ be the lengths of the opposite side, adjacent side, and hypoteneuse, respectively. We use these numbers to define the following functions:

$$\sin \theta = \frac{o}{h}, \quad \cos \theta = \frac{a}{h}, \quad \tan \theta = \frac{o}{a} = \frac{\sin \theta}{\cos \theta}, \quad \csc \theta = \frac{h}{o} = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{h}{a} = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{a}{o} = \frac{1}{\tan \theta}.$$
Some “special” angles have particularly nice trigonometric properties; we can use a unit circle (a circle of radius 1) to determine the trigonometric values for such angles.

First, let’s consider the special angle $\theta = \frac{\pi}{6}$:

We can determine third angle in the right triangle above, since the sum of the degrees in the angles of any triangle must be $\pi$ rad. The third angle is

$$\alpha = \pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}.$$ 

Now that we know values for all of its angles, let’s inspect the triangle above more closely; we would like to find values for $\sin\left(\frac{\pi}{6}\right)$, $\cos\left(\frac{\pi}{6}\right)$, etc. To do so, we must determine the lengths of each of the sides; fortunately, we know that the length of the hypotenuse is 1 since the triangle was embedded in a unit circle. Let’s “double” the triangle, as depicted below:

This new larger triangle is equiangular (all of its angles are $\frac{\pi}{3}$), thus is also equilateral—all of its sides have length 1. It is clear that the length of the opposite side is $o = \frac{1}{2}$, and we can use the Pythagorean identity $o^2 + a^2 = h^2$ to see that $a = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$.

So we have

$$\sin\left(\frac{\pi}{6}\right) = \frac{o}{h} = \frac{1}{2}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{a}{h} = \frac{\sqrt{3}}{2}, \quad \text{and} \quad \tan\left(\frac{\pi}{6}\right) = \frac{o}{a} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$
Let’s do the same thing for \( \theta = \frac{\pi}{4} \). The unit circle and right triangle for this case are graphed below:

![Unit Circle and Right Triangle for \( \theta = \frac{\pi}{4} \)](image)

It is clear that the remaining angle is \( \frac{\pi}{4} \); since two angles of the triangle are equal, the opposite sides are as well; thus \( a = o \). By the Pythagorean identity, \( a^2 + o^2 = h^2 \), which we rewrite as \( a^2 + a^2 = 1 \) or \( 2a^2 = 1 \). So \( a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \). Then

\[
\sin \frac{\pi}{4} = \frac{o}{h} = \frac{1}{\sqrt{2}}, \quad \cos \frac{\pi}{4} = \frac{a}{h} = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \tan \frac{\pi}{4} = \frac{o}{a} = 1.
\]

I strongly recommend that you memorize the following table of trig values at special angles in the first quadrant:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>0</td>
<td>( \sqrt{3} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>undefined</td>
</tr>
</tbody>
</table>

Our current definitions for the six trigonometric functions only apply to acute angles, i.e. angles less than \( \frac{\pi}{2} \); however, we may extend the definitions to apply to any angle.

Given an angle \( \theta \) in the \( xy \) plane, we may draw a ray from the origin in the appropriate direction, pick a point \((x, y)\) on the ray, and label the segment from the origin to \((x, y)\) with \( r \):
Then we define the trigonometric functions for angle $\theta$ as follows:

\[
\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \tan \theta = \frac{y}{x} \\
\csc \theta = \frac{r}{y} \quad \sec \theta = \frac{r}{x} \quad \cot \theta = \frac{x}{y}
\]

If $\theta < \frac{\pi}{2}$, then these definitions are precisely the same as the earlier definitions; these newer definitions simply allow us to apply the trigonometric functions to angles greater than $\frac{\pi}{2}$.

Note that the definitions tell us the signs of each of the trig functions in the different quadrants:

Another helpful set of facts to have at our disposal involves the definitions of the trig functions on the unit circle. Since $h = 1$, it is easy to check the accuracy of the following diagram:
Finding the values of trig functions for values of $\theta$ that do not lie in the first quadrant is made much simpler by using reference angles, which allow us to return to acute angles.

For example, consider finding $\sin \theta$ in the unit circle ($r = 1$) below:

In this example, $\sin \theta = \frac{y}{r} = y$. Now consider $\sin \alpha$:

It turns out that $\sin \alpha = y$ as well; in fact, if $\theta + \alpha = \pi$, then $\sin \theta = \sin \alpha$ for any $\frac{\pi}{2} < \theta \pi$. Since it is easier to evaluate trig functions on acute angles, we would really prefer to work with $\alpha$, and we call $\alpha$ a reference angle for $\theta$. 

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We can actually use a similar process for each of the trig functions in each of the four quadrants; the reference angle $\alpha$ in each quadrant is graphed below:

**Quadrant 2**

**Quadrant 3**

**Quadrant 4**
To determine the value for $\sin \theta$, $\cos \theta$, or $\tan \theta$, simply evaluate the trig function on the reference angle $\alpha$, then change the sign of the answer according to whether the function is positive or negative on the quadrant in which $\theta$ lies.

For example, let’s find $\sin \frac{4\pi}{3}$, $\cos \frac{4\pi}{3}$, and $\tan \frac{4\pi}{3}$:

The reference angle for $\theta = \frac{4\pi}{3}$ is

$$\alpha = \pi - \frac{4\pi}{3} = \frac{\pi}{3}.$$

In addition, the sine and cosine functions are negative in the third quadrant, whereas the tangent function is positive. Since

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{\pi}{3} = \frac{1}{2}, \quad \text{and} \quad \tan \frac{\pi}{3} = \sqrt{3},$$

we see that

$$\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}, \quad \cos \frac{4\pi}{3} = -\frac{1}{2}, \quad \text{and} \quad \tan \frac{4\pi}{3} = \sqrt{3}.$$
Finally, here are a few important identities to keep in mind when working with trig functions:

\[
\sin^2 \theta + \cos^2 \theta = 1, \quad \sec^2 \theta = \tan^2 \theta + 1, \quad \csc^2 \theta = \cot^2 \theta + 1
\]

\[
\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta
\]

\[
\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
\]

\[
\cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}
\]

\[
\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2}
\]