Section 7.1
Advanced Integration Techniques: Integration by Parts

In Calculus 1, we learned that we may differentiate the product of two functions by using the product rule:

\[ \frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x). \]

Unfortunately, finding the integral of a product is not so straightforward. However, we will on occasion be able to use a technique known as integration by parts when attempting to integrate certain types of products.

Let’s look at a particular example: set \( f(x) = x \) and \( g(x) = \sin x \). Then

\[ \frac{d}{dx}x \sin x = \sin x + x \cos x. \]

We could use this to our advantage if we wished to integrate the function \( x \cos x \): since

\[ x \cos x = \frac{d}{dx}x \sin x - \sin x, \]
we know that

\[
\int x \cos x \, dx = \int \left( \frac{d}{dx}x \sin x - \sin x \right) \, dx
\]
\[ = x \sin x - \int \sin x \, dx
\]
\[ = x \sin x + \cos x + C. \]

The process outlined above actually works for many of functions: since

\[ \frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x), \]
we know that

\[
\int (f'(x)g(x) + f(x)g'(x)) \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx
\]
\[ = f(x)g(x). \]

So

\[
\int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx = f(x)g(x)
\]
\[ \Rightarrow \int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx. \quad (1) \]

In other words, the integral of the product of functions \( f(x) \) and \( g'(x) \) can be rewritten as the sum of \( f(x)g(x) \) and the integral of \( f'(x)g(x) \). At first glance this may not seem particularly useful,
but it often occurs (as it did in the example above) that the integral of the function $f'(x)g(x)$ is much simpler to calculate than the integral of the original function $f(x)g'(x)$.

This observation is often written in the more convenient form

$$\int u \, dv = uv - \int v \, du. \quad (2)$$

The formula above corresponds to the formula in (1) when

$$u = f(x) \text{ and } dv = g'(x) \, dx,$$

so that

$$du = f'(x) \, dx \text{ (by differentiation) and } v = g(x) \text{ (by integration)}.$$ 

In particular, since we will need to evaluate the integral of the function $dv$, we must choose $dv$ so that it is a function whose integral we know.

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**Example.** Find $\int x \cos x \, dx$.

Unfortunately, we do not know an antiderivative of the function $x \cos x$, nor would rewriting the function be particularly helpful. We might be tempted to try integrating by u-substitution, but it should quickly become clear that there are no good choices for substitutions; if we try setting $u = \cos x$, then $du = -\sin x \, dx$, which is singularly unhelpful; nor would the substitution $u = x \cos x$ be useful.

So integration by parts is our only available option: of course, we looked at this example above, but will now demonstrate how to find the “parts” needed to apply our new integration technique. Recall that the formula is

$$\int u \, dv = uv - \int v \, du.$$

We need to make a choice for $u$ and a choice for $dv$. Let’s try setting

$$u = x \text{ and } dv = \cos x \, dx.$$ 

Then we have

$$u = x \quad \text{and} \quad dv = \cos x \, dx$$

$$du = dx \text{ (by differentiation)} \quad \text{and} \quad v = \sin x \text{ (by integration)}.$$ 

Then formula (2) tells us that

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx.$$ 

Fortunately for us, the last integral is easy to evaluate:

$$\int \sin x \, dx = -\cos x + C.$$
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So

\[ \int x \cos x \, dx = x \sin x - \int \sin x \, dx \]
\[ = x \sin x + \cos x + C. \]

To be certain that our answer is correct, we should differentiate the result:

\[ \frac{d}{dx} (x \sin x + \cos x + C) = \sin x + x \cos x - \sin x \]
\[ = x \cos x, \]

which confirms our calculation.

Notice what would have happened if we had switched our choices for \( u \) and \( dv \); we would have had

\[ u = \cos x \quad dv = x \, dx \]
\[ du = \sin x \, dx \quad (\text{by differentiation}) \quad v = \frac{1}{2} x^2 \quad (\text{by integration}). \]

Then we would have

\[ \int x \cos x \, dx = \frac{1}{2} x^2 \cos x - \frac{1}{2} \int x^2 \sin x \, dx. \]

Now

\[ \frac{1}{2} \int x^2 \sin x \, dx \]

is more complicated than was the original integral; so integration by parts did not simplify our problem but actually made it harder! In such a case, we should return to the original problem and try another approach. In general, when using integration by parts, we should attempt to choose \( u \) and \( dv \) so that the resultant integral \( \int v \, du \) is simpler than was the original integral.

**Example.** Find \( \int_{4}^{9} \frac{\ln y}{\sqrt{y}} \, dy \).

Let’s concentrate on finding the indefinite integral before we evaluate the definite integral. We will need to use integration by parts, and since we know how to integrate

\[ \int \frac{1}{\sqrt{y}} \, dy, \]

but *not*

\[ \int \ln y \, dy, \]

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let’s set

\[ u = \ln y \quad \text{ dv } = \frac{1}{\sqrt{y}} \, dy = y^{-1/2} \, dy \]
\[ du = \frac{1}{y} \, dy \text{ (by differentiation)} \quad v = 2y^{1/2} \text{ (by integration)}. \]

Then

\[
\int \frac{\ln y}{\sqrt{y}} \, dy = 2y^{1/2} \ln y - 2 \int y^{1/2} \, dy \\
= 2y^{1/2} \ln y - 2 \int y^{-1/2} \, dy \\
= 2y^{1/2} \ln y - 4y^{1/2} + C.
\]

Returning to the definite integral, we have

\[
\int_{4}^{9} \frac{\ln y}{\sqrt{y}} \, dy = 2y^{1/2} \ln y - 4y^{1/2} \bigg|_{4}^{9} \\
= 6 \ln 9 - 12 - (4 \ln 4 - 8) \\
= 6 \ln 3^2 - 4 \ln 2^2 - 4 \\
= 12 \ln 3 - 8 \ln 2 - 4.
\]

Example. Evaluate \( \int \ln x \, dx \).

Once again, we do not know an antiderivative of \( \ln x \), nor would rewriting the function or using a u-substitution help, so integration by parts is our only other choice. Since we know how to differentiate \( \ln x \), try

\[ u = \ln x \quad \text{ dv } = dx \]
\[ du = \frac{1}{x} \, dx \text{ (by differentiation)} \quad v = x \text{ (by integration)}. \]

Then

\[
\int \ln x \, dx = \int x \ln x - \int x \cdot \frac{1}{x} \, dx \\
= \int x \ln x - 1 \, dx \\
= x \ln x - x + C.
\]

Again, we may check our answer by differentiation:
\[
\frac{d}{dx}(x \ln x - x + C) = \ln x + \frac{x}{x} - 1 = \ln x.
\]

**Exercise:**
Show that \( \int x^2 e^x = e^x(x^2 - 2x + 2) \).

**Example.** Find \( \int e^{2\theta} \sin(3\theta) \, d\theta \).

Again, it quickly becomes clear that our only possible option for integrating the function is integration by parts. We would probably like to break up the function as \( e^{2\theta} \) and \( \sin(3\theta) \), but which of the two should be \( u \) and which should be \( dv \)? Separately, both of the components are easy to integrate, so there does not seem to be an obvious way to make the choice. Let’s try

\[
u = e^{2\theta}\]
\[
dv = \sin(3\theta) \, d\theta \]
\[
\quad du = 2e^{2\theta} \, d\theta \quad \text{(by differentiation)}
\]
\[
v = -\frac{1}{3} \cos(3\theta) \quad \text{(by integration).}
\]

Then
\[
\int e^{2\theta} \sin(3\theta) \, d\theta = -\frac{1}{3} e^{2\theta} \cos(3\theta) - \int \left( -\frac{2}{3} \right) e^{2\theta} \cos(3\theta) \, d\theta
\]
\[
= -\frac{1}{3} e^{2\theta} \cos(3\theta) + \frac{2}{3} \int e^{2\theta} \cos(3\theta) \, d\theta.
\]

Now we need to evaluate the new integral above, which is no simpler than the integral with which we started! However, it is no more complicated either (if it had been more complicated, we would probably want to start over and try a different approach). Let’s try evaluating
\[
\frac{2}{3} \int e^{2\theta} \cos(3\theta) \, d\theta
\]
using a second integration by parts. When choosing the parts, we should choose them carefully, corresponding to the original choices for parts above:

\[
u = e^{2\theta}\]
\[
dv = \cos(3\theta) \, d\theta \]
\[
\quad du = 2e^{2\theta} \, d\theta \quad \text{(by differentiation)}
\]
\[
v = \frac{1}{3} \sin(3\theta) \quad \text{(by integration).}
\]
Thus
\[
\frac{2}{3} \int e^{2\theta} \cos(3\theta) \, d\theta = \frac{2}{3} \left( \frac{1}{3} e^{2\theta} \sin(3\theta) - \int \frac{2}{3} e^{2\theta} \sin(3\theta) \, d\theta \right)
\]
\[
= \frac{2}{9} e^{2\theta} \sin(3\theta) - \frac{4}{9} \int e^{2\theta} \sin(3\theta) \, d\theta.
\]

We seem to be going in circles, since we have now returned to \( \int e^{2\theta} \sin(3\theta) \, d\theta \)! To summarize, here is what we have determined:

\[
\int e^{2\theta} \sin(3\theta) \, d\theta = -\frac{1}{3} e^{2\theta} \cos(3\theta) + \frac{2}{3} \int e^{2\theta} \cos(3\theta) \, d\theta
\]
\[
= -\frac{1}{3} e^{2\theta} \cos(3\theta) + \frac{2}{9} e^{2\theta} \sin(3\theta) - \frac{4}{9} \int e^{2\theta} \sin(3\theta) \, d\theta.
\]

As we noted earlier, the unknown term \( \int e^{2\theta} \sin(3\theta) \, d\theta \) shows up on both sides. What happens if we add \( \frac{4}{9} \int e^{2\theta} \sin(3\theta) \, d\theta \) to both sides of the equality, in effect “solving” for the unknown? We end up with

\[
\int e^{2\theta} \sin(3\theta) \, d\theta + \frac{4}{9} \int e^{2\theta} \sin(3\theta) \, d\theta = -\frac{1}{3} e^{2\theta} \cos(3\theta) + \frac{2}{9} e^{2\theta} \sin(3\theta) - \frac{4}{9} \int e^{2\theta} \sin(3\theta) \, d\theta + \frac{4}{9} \int e^{2\theta} \sin(3\theta) \, d\theta
\]
\[
= -\frac{1}{3} e^{2\theta} \cos(3\theta) + \frac{2}{9} e^{2\theta} \sin(3\theta).
\]

Recall that we wanted to determine \( \int e^{2\theta} \sin(3\theta) \, d\theta \); multiplying both sides of the last equation above by \( \frac{9}{13} \), we see that

\[
\int e^{2\theta} \sin(3\theta) \, d\theta = -\frac{3}{13} e^{2\theta} \cos(3\theta) + \frac{2}{13} e^{2\theta} \sin(3\theta) + C.
\]

In general, a function that is a product of two of the functions \( e^{kx}, \cos kx, \) and \( \sin kx \) can be integrated by using integration by parts twice, then solving for the integral.

Some problems may require multiple techniques in order to solve them.

**Example.** Find \( \int t \sec^{-1} t \, dt \).

Integration by parts seems natural here, and since we do not know how to evaluate

\[
\int \sec^{-1} t \, dt,
\]
we will start by setting

\[ u = \sec^{-1} t \quad \text{and} \quad dv = t \, dt \]

\[ du = \frac{1}{t\sqrt{t^2 + 1}} \, dt \quad \text{(by differentiation)} \quad v = \frac{1}{2} t^2 \quad \text{(by integration)}. \]

Then

\[ \int t \sec^{-1} t \, dt = \frac{1}{2} t^2 \sec^{-1} t - \frac{1}{2} \int \frac{t}{\sqrt{t^2 - 1}} \, dt. \]

Unfortunately, we don’t know how to integrate

\[ \int \frac{t}{\sqrt{t^2 - 1}} \, dt \]

immediately. We should try using a substitution to evaluate this integral: setting \( w = t^2 - 1 \), we have

\[ dw = 2t \, dt \quad \text{so that} \quad \frac{1}{2} \, dw = t \, dt. \]

So

\[
\frac{1}{2} \int \frac{t}{\sqrt{t^2 - 1}} \, dt = -\frac{1}{4} \int \frac{1}{\sqrt{w}} \, dw \\
= -\frac{1}{4} \int w^{-1/2} \, dw \\
= -\frac{1}{4} w^{1/2} + C \\
= -\frac{1}{2} (t^2 - 1)^{1/2} + C \\
= -\frac{1}{2} \sqrt{t^2 - 1} + C.
\]

Thus

\[
\int t \sec^{-1} t \, dt = \frac{1}{2} t^2 \sec^{-1} t - \frac{1}{2} \int \frac{t}{\sqrt{t^2 - 1}} \, dt \\
= \frac{1}{2} t^2 \sec^{-1} t - \frac{1}{2} \sqrt{t^2 - 1} + C.
\]

**Exercise:**

Evaluate \( \int e^{\sqrt{3s + 9}} \, ds \) by starting with a \( u \)-substitution.

**Hint:** Start with the substitution \( w = \sqrt{3s + 9} \).