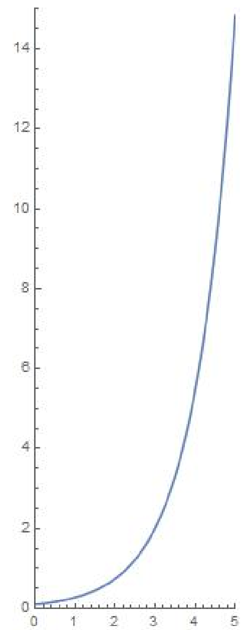
Calculus 1 Review

Limits

We can think of a limit as a way to predict a function's behavior at a point by looking at its behavior *near* the point.

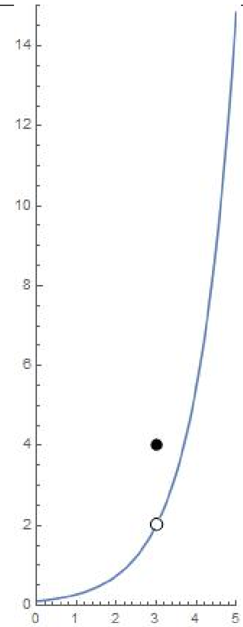
The way that a function $f(x)$ behaves *near* the point a may or may not be indicative of how the function behaves *at* the point a; if $f(x)$ is a "nice" function, then I can tell you what $f(a)$ is just by looking at the behavior of $f(x)$ *near* a. Such a function is called *continuous.*

The function below is continuous:



For example, if we were to predict the function's behavior at $x = 3$ by looking at inputs *near* $x = 3$, we would guess that the function's value was 2. The function behaves exactly as we would expect it to; symbolically, we write $\lim_{x\to 3} f(x) = f(3)$.

However, the function below is *not* continuous:



If we try to predict this function's behavior at $x = 3$, we would again guess that the function's value was 2. However, this is *not* how the function behaves; in fact, $f(3) = 4$. So $\lim_{x\to 3} f(x) \neq f(3)$.

Nearly all of the functions that we study in calculus are continuous on their domains. Practically speaking, this means that, if the number a is in the domain of $f(x)$, then

$$\lim_{x\to a} f(x) = f(a);$$

this rule, known as the *evaluation rule*, is just another way of saying that the function's behavior *near* the number a (i.e. $\lim_{x\to a} f(x)$) is the same as its behavior *at* the number a (i.e., $f(a)$).

Example. Find $\lim_{x\to 0} \dfrac{x^3 - 3x + 2}{x + 1}$.

Since rational functions are continuous on their domains, and 0 is in the domain of $\frac{x^3-3x+2}{x+1}$, we can use the evaluation rule to show that

$$\begin{aligned} \lim_{x\to 0} \frac{x^3 - 3x + 2}{x + 1} &= \frac{0^3 - 3 \cdot 0 + 2}{0 + 1} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

If $f(x)$ is a continuous function, but the number a is not in the domain of f, then we cannot use the evaluation rule to find the limit. Instead, we have a collection of ad-hoc rules and techniques to help determine the limit.

1. Use the Squeeze Theorem.
2. If examination of the function at $x = a$ results in an expression of the form $\frac{t}{0}$, where t is a non-zero real number, the limit does not exist.
3. If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, the limit does not exist.
4. If examination of the function at $x = a$ results in an expression of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the limit may or may not exist:
   (a) If the function can be rewritten by canceling common factors, do so, then reevaluate the limit.
   (b) If the function can be rewritten by rationalizing and canceling common factors or by adding fractions, do so, then reevaluate the limit.

Example. Find $\lim_{x\to 3} \frac{x^2-9}{x^2-6x+9}$.

Notice that, while the function is continuous on its domain, $x = 3$ is *not* in the domain; in particular, examination of the function at $x = 3$ yields an expression of the form $\frac{0}{0}$, so the limit may or may not exist.

Let's try factoring the function:

$$\begin{aligned} \frac{x^2-9}{x^2-6x+9} &= \frac{(x-3)(x+3)}{(x-3)^2} \\ &= \frac{(x+3)}{(x-3)} \qquad \text{when } x \neq 3. \end{aligned}$$

In particular,

$$\lim_{x\to 3} \frac{x^2-9}{x^2-6x+9} = \lim_{x\to 3} \frac{x+3}{x-3}.$$

But we cannot evaluate the second limit either, since 3 is not in the domain of $\frac{x+3}{x-3}$. This time, examination of the function at $x = 3$ yields an expression of the form $\frac{6}{0}$, and we may immediately conclude that the limit does not exist.

Technically, we should determine if the limit could be pinned down more precisely as ∞ or $-\infty$, but for now it is enough to realize the the limit does not exist.

Derivatives

Given a function $f(x)$, we might like to know the rate at which the outputs $f(x)$ change with respect to the inputs x. The derivative $f'(x)$ of $f(x)$ answers the question.

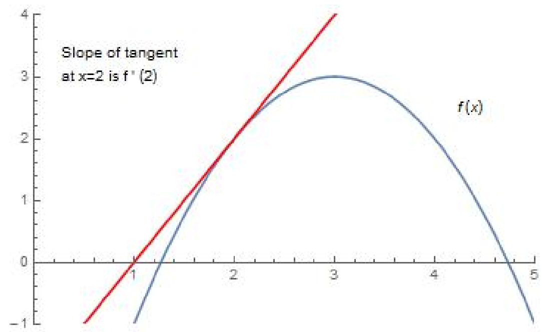
Definition. The derivative of the function $f(x)$ is the function $f'(x)$ given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

$f'(x)$ is the function built from $f(x)$ that tells us:

- the instantaneous rate of change of $f(x)$ at any point (assuming this makes sense)
- the slope of a tangent line drawn to $f(x)$ at any point to which a tangent can be drawn.



Fortunately, we do not have to calculate derivatives using the limit definition; the following "shortcut" rules for differentiating basic functions hide limits in the background (note that c, a, and n are constants):

function	derivative	function	derivative
c	0	$(n \neq 0)$ x^n	nx^{n-1}
a^x	$a^x \ln a$	$\sin x$	$\cos x$
$\cos x$	$-\sin x$	$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$	$\ln x$	$\frac{1}{x}$

If we need to differentiate a function that is a product, quotient, or composition of some combination of the aforementioned "basic" functions, we will need to employ the appropriate rule:

Product Rule:

If $f(x)$ and $g(x)$ have derivatives $f'(x)$ and $g'(x)$ respectively, then

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule:

If $f(x)$ and $g(x)$ have derivatives $f'(x)$ and $g'(x)$ respectively, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Chain Rule:

If $f(x)$ and $g(x)$ have derivatives $f'(x)$ and $g'(x)$ respectively, then

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Example. Find the derivative of $h(x) = 3^{x \ln x}$.

Notice that this particular function is not a basic function from the list above; that means that we do not automatically know its derivative. Instead, the function is made up of several different basic functions, which were combined by composition. So we will need to use the chain rule to differentiate $h(x)$. The chain rule says that we must break the function into "inside" and "outside" functions. We can rewrite by setting $g(x) = x \ln x$ (the "inside") and $f(x) = 3^x$ (the "outside"). Then

$$f(g(x)) = f(x \ln x) = 3^{x \ln x} = h(x).$$

Now that we have determined the inside and outside functions, we may apply the chain rule: since

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x),$$

let's calculate $f'(x)$ and $g'(x)$. First of all,

$$f(x) = 3^x, \text{ so that } f'(x) = 3^x \ln 3.$$

Now $g(x) = x \ln x$ is *not* a basic function–it is actually a product of the two functions x and $\ln x$, so we must use the product rule to find $g'(x)$. The product rule says that

$$g(x) = x \cdot \ln x, \text{ so that } g'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Finally, we can apply the chain rule:

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\ &= (3^{x \ln x} \ln 3)(\ln x + 1).\end{aligned}$$

So $h'(x) = (3^{x \ln x} \ln 3)(\ln x + 1)$.

Integrals

An antiderivative of a function $f(x)$ is another function $F(x)$ so that $F'(x) = f(x)$. In other words, if we wish to find an antiderivative of $f(x)$, we need to locate a function $F(x)$ whose derivative is $f(x)$. Antiderivatives simply reverse the process of differentiation.

$$f(x) \underset{\text{find derivative}}{\overset{\text{find antiderivative}}{\rightleftarrows}} F(x)$$

Definition. The set of all antiderivatives of $f(x)$ is called the indefinite integral of $f(x)$ with respect to x, and is denoted

$$\int f(x)dx.$$

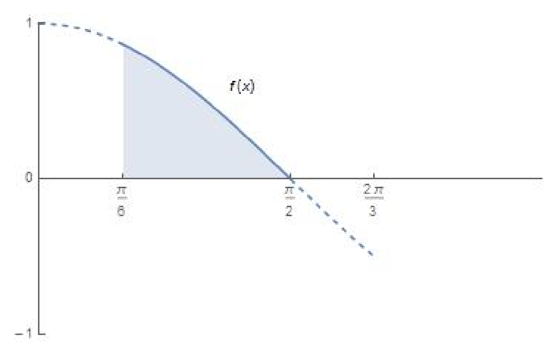
If $F(x)$ is any antiderivative of $f(x)$, then $\int f(x)dx = F(x) + C$.

We also studied areas in Calculus 1:

Definition. The definite integral $\displaystyle\int_a^b f(x)dx$ denotes

1. The area under $f(x)$ from a to b if $f(x) \geq 0$ on $[a, b]$;
2. The difference between areas above the x-axis and areas below the x-axis if $f < 0$ on $[a, b]$.

For example, consider the function $f(x)$ graphed below:



The quantity

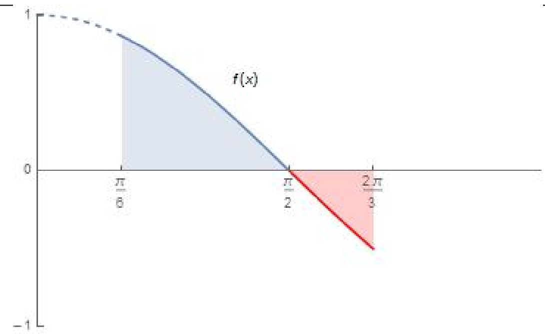
$$\int_{\pi/6}^{\pi/2} f(x)\ \mathrm{d}x$$

is exactly the area of the region shaded above in blue.

On the other hand, the quantity

$$\int_{\pi/6}^{2\pi/3} f(x)\ \mathrm{d}x$$

is actually the *difference between* the area of the blue shaded region and the area of the red shaded region below:



The Second Part of the Fundamental Theorem of Calculus gives us a meaningful way to calculate definite integrals and ties together the ideas of definite and indefinite integrals:

Theorem. If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

In other words, to calculate the definite integral of $f(x)$ from $x = a$ to $x = b$, first calculate the indefinite integral $\int f(x) = F(x) + C$, then plug b and a into $F(x)$ and evaluate $F(b) - F(a)$.

In summary, we know know that if $F(x)$ is an antiderivative of $f(x)$, then:

(indefinite integral) $\qquad \int f(x)dx = F(x) + C$

(definite integral) $\qquad \int_a^b f(x)dx = F(b) - F(a).$

The following table lists the rules for evaluating indefinite integrals of basic functions. Note that it is simply the reverse of the derivative table listed above:

function	integral	function	integral
x^n $(n \neq -1)$	$\frac{1}{n+1}x^{n+1} + C$	e^x	$e^x + C$
$\cos x$	$\sin x + C$	$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$	$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$	$\csc^2 x$	$-\cot x + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	a^x	$\frac{1}{\ln a}a^x + C$

Of course, there are many integrals which cannot be evaluated by simply looking at our table of integrals; in such a case, we must employ an advanced integration technique. In Calculus 1, we learned an advanced integration technique called u-substitution:

Theorem 6.3.4. If $u = g(x)$ is a differentiable function, $f(x)$ is continuous, and $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(g(x))g'(x)dx = \int f(u)du = F(g(x)) + C.$$

To use u-substitution, we will need to:

1. identify the inside function $g(x)$,
2. replace $g(x)$ and $g'(x)$ by the variable u and its derivative $\mathrm{d}u$,
3. evaluate the rewritten integral with respect to u, and
4. return to the original variable x.

Example. Find $\int x \csc(x^2) \cot(x^2)\, \mathrm{d}x$.

We cannot evaluate the integral directly, since we do not know any function whose derivative is $x \csc(x^2) \cot(x^2)$. So our next step is to try to use u-substitution. We want to replace a portion of $x \csc(x^2) \cot(x^2)$ with the variable u.

Of course, a large part of the reason that this integral is difficult is that the input for the trig functions is x^2, and not x. Thus it seems reasonable to replace x^2; if we let $u = x^2$, then differentiating both sides of the relationship with respect to x gives us

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \text{ or } \mathrm{d}u = 2x\ \mathrm{d}x.$$

Notice however, that $2x\ \mathrm{d}x$ does not appear in the integrand–instead, the integrand involves a factor of $x\ \mathrm{d}x$. Fortunately, this is not a problem; we simply rewrite

$$\mathrm{d}u = 2x\ \mathrm{d}x \text{ as } \frac{1}{2}\ \mathrm{d}u = x\ \mathrm{d}x.$$

So we rewrite the integral as

$$\int x \csc(x^2) \cot(x^2) dx = \frac{1}{2} \int \csc u \cot u\ \mathrm{d}u.$$

The second integral is much easier to evaluate:

$$\frac{1}{2} \int \csc u \cot u\ \mathrm{d}u \quad = \quad -\frac{1}{2} \csc u + C.$$

Returning to the original variable via $u = x^2$, we have

$$\begin{aligned} \frac{1}{2} \int \csc u \cot u\ \mathrm{d}u &= -\frac{1}{2} \csc u + C \\ &= -\frac{1}{2} \csc(x^2) + C, \end{aligned}$$

so that

$$\int x \csc(x^2) \cot(x^2)\ \mathrm{d}x = -\frac{1}{2} \csc(x^2) + C.$$