Orthonormality and the Gram-Schmidt Process

The basis

$$(e_1,\ e_2,\ \ldots,\ e_n)$$

for $\mathbb{R}^n$ (or $\mathbb{C}^n$) is considered the *standard* basis for the space because of its geometric properties under the standard inner product:

1. $||e_i|| = 1$ for all i, and
2. e_i, e_j are orthogonal whenever $i \neq j$.

With this idea in mind, we record the following definitions:

Definitions 6.23/6.25. Let V be an inner product space.

- A list of vectors in V is called *orthonormal* if each vector in the list has norm 1, and if each pair of distinct vectors is orthogonal.
- A basis for V is called an *orthonormal basis* if the basis is an orthonormal list.

Remark. If a list $(v_1,\ \ldots,\ v_n)$ is orthonormal, then

$$\langle v_i,\ v_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Example. The list

$$(e_1,\ e_2, \ldots,\ e_n)$$

forms an orthonormal basis for $\mathbb{R}^n/\mathbb{C}^n$ under the standard inner products on those spaces.

Example. The standard basis for $\mathcal{M}_n(\mathbb{C})$ consists of n^2 matrices e_{ij}, $1 \leq i, j \leq n$, where e_{ij} is the $n \times n$ matrix with a 1 in the ij entry and 0s elsewhere. Under the standard inner product on $\mathcal{M}_n(\mathbb{C})$ this is an orthonormal basis for $\mathcal{M}_n(\mathbb{C})$:

1. $\langle e_{ij}, e_{ij}\rangle$:

$$\begin{aligned} \langle e_{ij}, e_{ij}\rangle &= \operatorname{tr}(e_{ij}^* e_{ij}) \\ &= \operatorname{tr}(e_{ji} e_{ij}) \\ &= \operatorname{tr}(e_{jj}) \\ &= 1. \end{aligned}$$

2. $\langle e_{ij}, e_{kl}\rangle$, $k \neq i$ or $j \neq l$:

$$\begin{aligned}\langle e_{ij}, e_{kl}\rangle &= \operatorname{tr}(e_{kl}^* e_{ij}) \\ &= \operatorname{tr}(e_{lk} e_{ij}) \\ &= \operatorname{tr}(\mathbf{0}) \text{ if } k \neq i, \text{ or } \operatorname{tr}(e_{lj}) \text{ if } k = i \text{ but } l \neq j \\ &= 0.\end{aligned}$$

So every vector in the list has norm 1, and every distinct pair of vectors is orthogonal.

Example. Show that the list

$$\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right)$$

forms an orthonormal basis for $\mathcal{M}_2(\mathbb{R})$.

We won't work through all of the details for this example; indeed, all of the calculations are similar. We start by verifying that each vector in the list has norm 1:

$$\begin{aligned} \left\| \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right\| &= \sqrt{\operatorname{tr}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right)} \\ &= \sqrt{\operatorname{tr} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}} \\ &= 1. \end{aligned}$$

Similarly, each vector in the list has norm 1.

Next, we should check that each pair of distinct vectors is orthogonal; we only check

$$\begin{aligned} \left\langle \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right) \\ &= \operatorname{tr} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Similarly, each pair of distinct vectors is orthogonal. Thus the basis is an orthonormal basis.

Orthonormal lists (and orthonormal bases in particular) have several important properties:

Theorem 6.26. Every orthonormal list of vectors is linearly independent.

Theorem 6.30. Let $(e_1,\ e_2,\ \ldots,\ e_n)$ be an orthonormal basis of V. Then for any $v \in V$,

$$v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 + \ldots + \langle v, e_n\rangle e_n.$$

Proof. Every $v \in V$ is a linear combination of basis vectors, say

$$v = a_1 e_1 + \ldots + a_n e_n;$$

since the basis vectors are orthonormal, we can easily calculate a_i by evaluating

$$\begin{aligned}\langle v, e_i\rangle &= \langle a_1 e_1 + \ldots + a_n v_n, e_i\rangle \\ &= a_1\langle e_1, e_i\rangle + \ldots + a_i\langle e_i, e_i\rangle + a_n\langle e_n, e_i\rangle \\ &= a_i\end{aligned}$$

(again, this follows because basis vectors are orthonormal).

Example. Find the coordinate vector for

$$v = \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}$$

with respect to the orthonormal basis

$$B = \left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right).$$

Notice that Theorem 6.30 makes this calculation significantly easier than it would have been otherwise. To find the coordinates for v with respect to B, we simply need to evaluate a few inner products:

$$\begin{aligned}\left\langle \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix} \right) \\ &= 4\sqrt{2};\end{aligned}$$

$$\begin{aligned}\left\langle \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix} \right) \\ &= -3\sqrt{2};\end{aligned}$$

$$\begin{aligned}\left\langle \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix} \right) \\ &= -2\sqrt{2};\end{aligned}$$

and

$$\left\langle \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right\rangle = \operatorname{tr}\left(\begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix} \right) = 3\sqrt{2}.$$

Thus

$$(v)_B = \sqrt{2} \begin{pmatrix} 4 \\ -3 \\ -2 \\ 3 \end{pmatrix}.$$

The Gram-Schmidt Procedure

Our final topic in this course is a powerful algorithm known as the Gram-Schmidt procedure. The algorithm gives us a method for finding an orthonormal basis for any finite dimensional space.

The Gram-Schmidt Procedure. Let $v_1, \ldots, v_m$ be a linearly independent list of vectors in V, and set

$$e_1 = \frac{v_1}{||v_1||}.$$

For $2 \leq j \leq m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}}{||v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}||}.$$

Then $e_1, \ldots, e_m$ is an independent list so that

$$\operatorname{span}(e_1, \ldots, e_m) = \operatorname{span}(v_1, \ldots, v_m).$$

The algorithm has an immediate corollary:

Corollary 6.34. Every finite dimensional inner product space has an orthonormal basis.

Example. Apply the Gram-Schmidt procedure to basis

$$B = (x^3 - x^2,\ x^2 - x,\ x - 1,\ 1)$$

of $\mathcal{P}_3(\mathbb{R})$ (under the standard inner product on $\mathcal{P}_3(\mathbb{R})$) to find an orthonormal basis for $\mathcal{P}_3(\mathbb{R})$.

Setting

$$\begin{aligned} v_1 &= x^3 - x^2 \\ v_2 &= x^2 - x \\ v_3 &= x - 1 \\ v_4 &= 1, \end{aligned}$$

let us begin by calculating $||v_1||$ under the standard inner product

$$\langle \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0 \rangle = \alpha_3\beta_3 + \alpha_2\beta_2 + \alpha_1\beta_1 + \alpha_0\beta_0,$$

we have

$$\begin{aligned} ||v_1|| &= \sqrt{\langle x^3 - x^2,\ x^3 - x^2 \rangle} \\ &= \sqrt{1+1} \\ &= \sqrt{2}. \end{aligned}$$

Thus we set

$$e_1 = \frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2.$$

Next, we find e_2 using the formula

$$e_2 = \frac{v_2 - \langle v_2,\ e_1 \rangle e_1}{||v_2 - \langle v_2,\ e_1 \rangle e_1||}:$$

$$\begin{aligned} v_2 - \langle v_2,\ e_1 \rangle e_1 &= x^2 - x - \langle x^2 - x,\ \frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2 \rangle (\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2) \\ &= x^2 - x + \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2) \\ &= x^2 - x + \frac{1}{2}x^3 - \frac{1}{2}x^2 \\ &= \frac{1}{2}x^3 + \frac{1}{2}x^2 - x; \end{aligned}$$

we also need

$$\begin{aligned} ||\frac{1}{2}x^3 + \frac{1}{2}x^2 - x|| &= \sqrt{\frac{1}{4} + \frac{1}{4} + 1} \\ &= \sqrt{\frac{3}{2}}. \end{aligned}$$

Thus

$$e_2 = \frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x.$$

Moving on, we find e_3 using the formula

$$e_3 = \frac{v_3 - \langle v_3,\ e_1 \rangle e_1 - \langle v_3,\ e_2 \rangle e_2}{||v_3 - \langle v_3,\ e_1 \rangle e_1 - \langle v_3,\ e_2 \rangle e_2||}:$$

$$\begin{aligned}
v_3 - \langle v_3,\ e_1\rangle e_1 - \langle v_3,\ e_2\rangle e_2 &= x - 1 - \langle x-1,\ \frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2\rangle(\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2) \\
&\quad -\langle x-1, \frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x\ \rangle(\frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x) \\
&= x - 1 - (0)(\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2) - (-\sqrt{\frac{2}{3}})(\frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x) \\
&= x - 1 + \frac{1}{3}x^3 + \frac{1}{3}x^2 - \frac{2}{3}x \\
&= \frac{1}{3}x^3 + \frac{1}{3}x^2 + \frac{1}{3}x - 1,
\end{aligned}$$

whose norm is

$$\begin{aligned}
||\frac{1}{3}x^3 + \frac{1}{3}x^2 + \frac{1}{3}x - 1|| &= \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + 1} \\
&= \sqrt{\frac{12}{9}} \\
&= \frac{2}{\sqrt{3}}.
\end{aligned}$$

Thus we set

$$e_3 = \frac{1}{2\sqrt{3}}x^3 + \frac{1}{2\sqrt{3}}x^2 + \frac{1}{2\sqrt{3}}x - \frac{\sqrt{3}}{2}.$$

Finally, we need to calculate e_4, using the formula

$$e_4 = \frac{v_4 - \langle v_4,\ e_1\rangle e_1 - \langle v_4,\ e_2\rangle e_2 - \langle v_4, e_3\rangle e_3}{||v_4 - \langle v_4,\ e_1\rangle e_1 - \langle v_4,\ e_2\rangle e_2 - \langle v_4, e_3\rangle e_3||}:$$

$$\begin{aligned}
v_4 - \langle v_4,\ e_1\rangle e_1 - \langle v_4,\ e_2\rangle e_2 - \langle v_4, e_3\rangle e_3 &= 1 - \langle 1,\ \frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2\rangle(\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2) \\
&\quad -\langle 1,\ \frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x\rangle(\frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x) \\
&\quad -\langle 1,\ \frac{1}{2\sqrt{3}}x^3 + \frac{1}{2\sqrt{3}}x^2 + \frac{1}{2\sqrt{3}}x - \frac{\sqrt{3}}{2}\rangle \\
&\quad (\frac{1}{2\sqrt{3}}x^3 + \frac{1}{2\sqrt{3}}x^2 + \frac{1}{2\sqrt{3}}x - \frac{\sqrt{3}}{2}) \\
&= 1 - (0)(\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2) \\
&\quad -(0)(\frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x) \\
&\quad -(-\frac{\sqrt{3}}{2})(\frac{1}{2\sqrt{3}}x^3 + \frac{1}{2\sqrt{3}}x^2 + \frac{1}{2\sqrt{3}}x - \frac{\sqrt{3}}{2}) \\
&= 1 + \frac{1}{4}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x - \frac{3}{4} \\
&= \frac{1}{4}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}.
\end{aligned}$$

We have

$$\begin{aligned}
||\frac{1}{4}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}|| &= \sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}} \\
&= \sqrt{\frac{4}{16}} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus we set

$$e_4 = \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2},$$

and the list

$$(\frac{1}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}x^2,\ \frac{1}{\sqrt{6}}x^3 + \frac{1}{\sqrt{6}}x^2 - \sqrt{\frac{2}{3}}x,\ \frac{1}{2\sqrt{3}}x^3 + \frac{1}{2\sqrt{3}}x^2 + \frac{1}{2\sqrt{3}}x - \frac{\sqrt{3}}{2},\ \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2})$$

is an orthonormal basis for $\mathcal{P}_3(\mathbb{R})$.