Existence of Eigenvalues

Now that we have introduced the concept of eigenvalues of a transformation, we should ask whether or not a transformation is *guaranteed* to have them. Indeed, there are certainly examples of operators which have no eigenvalues; we look at one such in the next example.

Example. Show that the only operator of the form $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_\theta(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v$$

that has an eigenvalue (up to integer multiples of π) is T_0.

Recall that $\lambda \in \mathbb{R}$ is an eigenvalue of T_θ if and only if it is also an eigenvalue of the matrix

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};$$

thus we search for solutions to the characteristic equation

$$\det(M_\theta - \lambda I) = 0.$$

Since

$$M_\theta - \lambda I = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix},$$

we have

$$\begin{aligned} \det(M_\theta - \lambda I) &= \det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} \\ &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= \cos^2\theta + \sin^2\theta - 2\lambda\cos\theta + \lambda^2 \\ &= \lambda^2 - 2\lambda\cos\theta + 1. \end{aligned}$$

Using the quadratic equation to search for solutions $\lambda \in \mathbb{R}$, we have

$$\frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2}.$$

Now since

$$\cos^2\theta \leq 1,$$

the quadratic equation has real solutions if and only if $\theta = 0$. Thus the operator $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ has no (real) eigenvalues for $\theta \neq n\pi$.

This result should not be particularly surprising–T_θ acts on a vector by rotating it by θ. Rotation only scales a vector if the angle of rotation is an integer multiple of π, thus there are no real eigenvalues for other values of θ.

Eigenvalues are extremely useful tools for understanding spaces and their operators, so we would like to find some conditions on V and $T : V \to V$ that will guarantee that T does have eigenvalues. We will explore that problem in this section, but must first introduce the helpful concept of polynomials applied to operators.

Polynomials Applied to Operators

Definition 5.16. Let $T : V \to V$ be a linear operator, and let $m > 0$ be an integer. We define powers of T as follows:

- T^m is the linear operator
$$T^m = \underbrace{T \dots T}_{m \text{ copies}}.$$
- T^0 is defined to be the identity T_I on V.
- If T is an invertible operator, so that T^{-1} exists, then we use the notation T^{-m} to indicate the operator $(T^{-1})^m$.

The definition should remind you of our definitions for powers of matrices; indeed, the definitions are virtually identical, and it is easy to show that identities such as

$$T^m T^n = T^{m+n} \text{ and } (T^m)^n = T^{mn}$$

hold for $m, n \geq 0$.

A definition for powers of an operator is useful in that it will allow us to apply polynomials to operators, as indicated in the next definition:

Definitions 5.17. Let V be a vector space over $\mathbb{F}$, and let $T : V \to V$ be a linear operator. Let $p \in \mathcal{P}(\mathbb{F})$ be a polynomial, say

$$p(x) = \alpha_n x^n + \dots + \alpha_1 x + \alpha_0,$$

where $x \in \mathbb{F}$. Then $p(T) : V \to V$ is the operator defined by

$$p(T) = \alpha_n T^n + \dots + \alpha_1 T + \alpha_0 T_I.$$

Recall that the product pq of a pair of polynomials is the polynomial defined by

$$pq(x) = p(x)q(x).$$

It is easy to show that

$$pq(T) = p(T)q(T) = q(T)p(T).$$

Example. Let

$$M = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

and define $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ by matrix multiplication by M, that is

$$T_M(v) = Mv.$$

Recall that M is a *rotation matrix*, that is, it rotates a given vector v by $\pi/6$ counterclockwise.

Let $p(x) = x^4 - 2x^3 - 2x$. Find the transformation $p(T_M)$.

Clearly

$$p(T_M) = T_M^4 - 2T_M^3 - 2T_M;$$

of course, we would like to describe the action of $p(T)$ on a given vector $v \in \mathbb{R}^2$. Clearly $T^k(v) = M^k v$, so we merely need to understand powers of M to make the calculation.

Since M is a rotation matrix, it will be quite simple to calculate the powers: indeed, M^k rotates a given vector by $k\pi/6$. Thus

$$\begin{aligned} M^4 &= \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} -2M^3 &= -2\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} -2M &= -2\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}. \end{aligned}$$

Thus the matrix corresponding to $p(T_M)$ is

$$\begin{pmatrix} -\sqrt{3}-\frac{1}{2} & 3-\frac{\sqrt{3}}{2} \\ -3+\frac{\sqrt{3}}{2} & -\sqrt{3}-\frac{1}{2} \end{pmatrix},$$

that is

$$p(T_M)v = \begin{pmatrix} -\sqrt{3}-\frac{1}{2} & 3-\frac{\sqrt{3}}{2} \\ -3+\frac{\sqrt{3}}{2} & -\sqrt{3}-\frac{1}{2} \end{pmatrix} v.$$

Existence of Eigenvalues

We are now ready to answer the question that motivated our discussion of polynomials on operators:

Which linear operators are guaranteed to have eigenvalues?

The next theorem answers the question:

Theorem 5.21. Let V be a finite dimensional nontrivial vector space over $\mathbb{C}$; then every linear operator on V has an eigenvalue.

Proof. Let V be an n dimensional vector space over $\mathbb{C}$, $n > 0$, and let $T : V \to V$ be a linear operator. For any $v \in V$, the vectors

$$v,\ T(v),\ T^2(v),\ \ldots,\ T^n(v)$$

cannot be independent (since there are $n+1$ of them), so there are some $\alpha_i \in \mathbb{C}$ not all 0 so that

$$\alpha_n T^n(v) + \ldots + \alpha_2 T^2(v) + \alpha_1 T(v) + \alpha_0 v = \mathbf{0}.$$

Let $p \in \mathcal{P}(\mathbb{C})$ be the polynomial defined by

$$p(x) = \alpha_n x^n + \ldots + \alpha_2 x^2 + \alpha_1 x + \alpha_0,$$

so that

$$p(T) = \alpha_n T^n + \ldots + \alpha_2 T^2 + \alpha_1 T + \alpha_0 T_I.$$

In particular,

$$p(T)(v) = \alpha_n T^n(v) + \ldots + \alpha_2 T^2(v) + \alpha_1 T(v) + \alpha_0 v = \mathbf{0}.$$

By the Fundamental Theorem of Algebra, $p(x)$ has a root λ; indeed by the division algorithm and repeated application of the Fundamental Theorem of Algebra, p has as many (not necessarily unique) roots as its degree, and factors as

$$p(x) = c(x-\lambda_1)\ldots(x-\lambda_m),$$

where c and each λ_i are elements of $\mathbb{C}$ ($m = n$ in case $\alpha_n \neq 0$).

Now

$$p(T) = c(T-\lambda_1 T_I)\ldots(T-\lambda_m T_I);$$

however,

$$\begin{aligned} p(T)(v) &= c(T - \lambda_1 T_I) \dots (T - \lambda_m T_I)(v) \\ &= \alpha_n T^n(v) + \dots + \alpha_2 T^2(v) + \alpha_1 T(v) + \alpha_0 v \\ &= \mathbf{0}. \end{aligned}$$

Thus at least one of the $T - \lambda_k T_I$ is not injective, which means that there is a nonzero $u \in V$ so that

$$(T - \lambda_k T_I)u = \mathbf{0};$$

equivalently,

$$T(u) = \lambda_k u.$$

Thus λ_k is an eigenvalue for T.

The theorem above cannot guarantee that an operator over $\mathbb{R}$ will have real eigenvalues; indeed, while many operators over $\mathbb{R}$ *do* have real eigenvalues, we have seen that $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_\theta(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v$$

does not, as long as $\theta \neq n\pi$.

However, if we think of T_θ as an operator over $\mathbb{C}^2$, that is $T_\theta : \mathbb{C}^2 \to \mathbb{C}^2$ again defined by

$$T_\theta(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v,$$

then T_θ is guaranteed to have eigenvalues (in $\mathbb{C}$). Indeed, the eigenvalues are given by

$$\begin{aligned} \lambda &= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2} \\ &= \cos\theta \pm i\sqrt{1 - \cos^2\theta} \\ &= \cos\theta \pm i\sin\theta \\ &= e^{i\theta} \text{ or } e^{-i\theta}. \end{aligned}$$

Upper Triangular Matrix for an Operator

Recall that, given an operator $T : V \to V$, there is a matrix $A_{(B,B)}$ with respect to *any* basis B for V so that

$$A_{(B,B)}(v)_B = (T(v))_B.$$

In particular, the matrices $A_{(B,B)}$ and $A_{(C,C)}$ are different if B and C are different bases. To make calculations easier, we prefer to find a matrix for our transformation that has a nice form, say diagonal or upper triangular. This amounts to finding the right basis for V.

It follows as a consequence of Theorem 5.21 that *every* operator on a finite dimensional vector space V over $\mathbb{C}$ has an upper triangular matrix. We record the theorem without proof:

Theorem 5.27. Let V be a finite dimensional vector space over $\mathbb{C}$, and let $T : V \to V$ be an operator. Then there is a basis B for V so that $A_{(B,B)}$ is upper triangular.

Example. The operator $T_\theta : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$T_\theta(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v$$

has matrix

$$A_{(B,B)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with respect to the standard basis B for $\mathbb{C}^2$. Find a basis C for V so that $A_{(C,C)}$ is upper triangular.

Since $\lambda = e^{i\theta}$ is an eigenvalue for T_θ, we know that the operator

$$T_\theta - e^{i\theta} T_I$$

has nontrivial nullspace, that is there is a vector $x \in \mathbb{C}^2$ so that

$$T_\theta(x) - e^{i\theta} T_I(x) = T_\theta(x) - e^{i\theta} v = \mathbf{0}.$$

It would be helpful to have one such eigenvector, so we calculate

$$\begin{aligned} \mathbf{0} &= T_\theta(v) - e^{i\theta} v \\ &= (T_\theta - e^{i\theta} T_I) v \\ &= \begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{i\theta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

Since $\cos\theta - e^{i\theta} = -i\sin\theta$, we may row reduce as follows:

$$\begin{aligned} \begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{i\theta} \end{pmatrix} &\to \begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ 0 & \cos\theta - e^{i\theta} + i\sin\theta \end{pmatrix} \\ &\to \begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Setting $x_2 = 1$, we se that

$$\begin{aligned}(\cos\theta - e^{i\theta})x_1 &= \sin\theta \\ x_1 &= \frac{\sin\theta}{\cos\theta - e^{i\theta}} \\ &= \frac{\sin\theta}{-i\sin\theta} \\ &= -\frac{1}{i} \\ &= i.\end{aligned}$$

Thus

$$v = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

is an eigenvector associated with eigenvalue $e^{i\theta}$. In particular, if we let $C = (v, v')$ (with v' as yet to be determined), then since

$$T(v) = e^{i\theta} v,$$

we know that

$$(T(v))_C = e^{i\theta}(v)_C = e^{i\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now since eigenvectors associated with unique eigenvalues are linearly independent by Theorem 5.10, we can easily complete C to a basis by using the eigenvector

$$v' = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

associated with eigenvalue $\lambda = e^{-i\theta}$. Thus our basis is

$$C = (v, v') = \left(\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right).$$

The matrix $A_{(C,C)}$ for T with respect to C is

$$A_{(C,C)} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

which is upper triangular as desired (and actually diagonal).