Test3

Thursday, April 28, 2016 1:14 PM

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Math 300

Test 3

April 29, 2016

Name: Key

You must show ALL of your work in order to receive credit.

If your scratch paper shows work that leads to your solution, please turn in in inside the test. Otherwise, dispose of it yourself.

1. Modified true/false: If the statement below is *always* true, write "true". Otherwise, correct the statement. (2 points each)

(a) Let A be an $n \times n$ matrix so that the matrix equation $(A - 3I)x = \mathbf{0}$ has only the trivial solution. Then 3 is not an eigenvalue for A.

True

(b) If $\lambda \in \mathbb{F}$ is an eigenvalue for the operator $T : V \to V$, then $T(v) = \lambda v$ for all $v \in V$.

False - for all eigenvectors $v \in V$

(c) If $A = A_{(B,C)}$ is the matrix for $T : V \to W$ with respect to bases B and C for V and W respectively, then $Av = T(v)$ for all $v \in V$.

False - $A(v)_B = (T(v))_C$

(d) If $T : V \to W$ is an injective transformation, then $\dim(V) = \dim(\text{range}(T))$.

True

(e) If V is 5 dimensional and W is 3 dimensional, then no transformation $T : V \to W$ can be surjective.

False - No transformation is injective.

2. Find the matrix $A = A_{(B,C)}$ for the transformation $T : U_2(\mathbb{R}) \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a+b \\ a-c \end{pmatrix}$$

with respect to bases

$$B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$C = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).$$

(15 points)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $$A = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3. The vector space $\mathcal{P}_1(\mathbb{R})$ of all linear polynomials over $\mathbb{R}$ has basis $B = (x-1, \ -x)$. The matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

is the transition matrix from a basis C for $\mathcal{P}_1(\mathbb{R})$ to basis B. Find C. (15 points)

$C = (v_1, v_2)$

$v_1 = 2(x-1) - (-x)$

$= 2x - 2 + x$

$v_1 = 3x - 2$

$v_2 = 1(x-1) + 1(-x)$

$= -1$

So $C = (3x-2, -1)$

4. The linear transformation $T : \mathbb{C}^3 \to \mathbb{C}$ is defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1 + 2x_2 + 3x_3.$$

Find a basis for the null space of T. (15 points)

If $x_1 + 2x_2 + 3x_3 = 0$

then $x_1 = -2x_2 - 3x_3$.

So every $x \in Null(T)$ has form $x = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

So one basis for null (T) is $\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right)$

5. Let $T : \mathcal{P}_1(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ be the transformation defined by $T(\alpha x + \beta) = \alpha(x+1) + \beta$. Find an eigenvalue λ for T and an eigenvector associated with λ. (15 points)

if $\lambda(\alpha x + \beta) = \alpha x + \alpha + \beta$

then $\lambda \alpha x = \alpha x \Rightarrow \lambda = 1$

but $\lambda \beta = \alpha + \beta \Rightarrow \beta(\lambda - 1) = \alpha$

so $\alpha = 0$.

Thus $\lambda = 1$ has associated eigenvector $p(x) = \beta$ for any $\beta \in \mathbb{R}$.

$\boxed{\lambda = 1,\ p(x) = 3}$ is a specific example.

6. The vector space $\mathfrak{sl}(2,\mathbb{C})$ of 2×2 trace 0 matrices has basis

$$(e_1,\ e_2,\ e_3) = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right).$$

Given $X, Y \in \mathfrak{sl}(2,\mathbb{C})$ with

$$X = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \text{ and } Y = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3,$$

we may define an inner product on $\mathfrak{sl}(2,\mathbb{C})$ by

$$\langle X, Y\rangle = \alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \alpha_3\overline{\beta_3}.$$

(a) Given $X = \begin{pmatrix} 1 & i \\ 1 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 1 \\ i & -2 \end{pmatrix}$, find $\langle X, Y\rangle$. (5 points)

$$\langle X, Y\rangle = 1\cdot 2 + i\cdot 1 + 1\cdot(-i)$$
$$= 2 + i - i = \boxed{2}$$

(b) Find $||X||$. (5 points)

$$||X|| = \sqrt{\langle X, X\rangle} = \sqrt{1 + i(-i) + 1}$$
$$= \boxed{\sqrt{3}}$$

(c) Find a nonzero vector orthogonal to X. (5 points)

$W := \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$; then $\langle X, W\rangle = 0 + i - i = 0$

so W is orthogonal to X.

7. Let $T : V \to W$ be a linear transformation, and let $S : W \to Y$ be an isomorphism. Prove that $\text{null}\,(S \circ T) = \text{null}\,(T)$. (15 points)

if $x \in \text{null}(T)$ then $T(x) = \vec{0}$,

and $S(T(x)) = S(\vec{0}) = \vec{0}$,

so $\text{null}(T) \subset \text{null}(S \circ T)$.

On the other hand, if $x \in \text{null}(S \circ T)$

then $S \circ T(x) = \vec{0} \Rightarrow S(T(x)) = \vec{0}$.

But S is injective, so $T(x) = \vec{0}$,

Thus $x \in \text{null}(T)$

$\Rightarrow \text{null}(S \circ T) \subset \text{null}(T)$.