1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be given by

$$T(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) = 2\alpha_1 x^3 + (\alpha_3 + \alpha_2)x + (\alpha_1 + \alpha_0).$$

   (a) Is $x^3 - 5x^2 + 3x - 6$ in $\operatorname{null}(T)$? Explain why/why not.

   Solution: No, because

$$T(x^3 - 5x^2 + 3x - 6) = 6x^3 - 4x^2 - 3 \neq \mathbf{0}.$$

   (b) Is $4x^3 - 4x^2$ in $\operatorname{null}(T)$? Explain why/why not.

   Solution: Yes, because

$$T(4x^3 - 4x^2) = (4 - 4)x = \mathbf{0}.$$

   (c) Is $8x^3 - x - 1$ in $\operatorname{range}(T)$? Explain why/why not.

   Solution: Yes, because

$$T(-x^3 + 4x - 5) = 8x^3 - x - 1.$$

   (d) Is $4x^3 - 3x^2 + 7$ in $\operatorname{range}(T)$? Explain why/why not.

   Solution: No, because no vector whose x^2 component has nonzero coefficient is in the range of T.

2. Given

$$M = \begin{pmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{pmatrix},$$

define $T_M : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

   (a) Find the rank of M.

   Solution: The RREF of M is

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{pmatrix};$$

   since this matrix has 2 leading 1s, its rank is 2.

   (b) Find a basis for the null space of T_M.

   Solution: Solutions to $Mx = \mathbf{0}$ may be parameterized as

$$x = s\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix};$$

   thus one choice for a basis for $\operatorname{null}(T_M)$ is

$$\left(\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}\right).$$

(c) Find a basis for the range of T_M.

Solution: Row reducing the augmented matrix for the system, we have

$$\begin{pmatrix} 3 & 2 & 11 & |v_1 \\ 2 & 1 & 8 & |v_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5 & | & 2v_2 - v_1 \\ 0 & 1 & -2 & | & 2v_1 - 3v_2 \end{pmatrix}.$$

This system is always consistent, so the range of T_M is all of $\mathbb{R}^2$; thus we may choose any basis we like for $\mathbb{R}^2$, say

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

(d) Verify the Fundamental Theorem for T_M.

Solution: The dimension of null $(T_M) = 1$, and dimension of range $(T_M) = 2$; we have

$$\dim(\text{null}\,(T_M)) + \dim(\text{range}\,(T_M)) = 3 = \dim(\mathbb{R}^3).$$

3. Define $T : \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ by

$$T(X) = X - X^\top.$$

(a) Find a basis for the null space of T.

Solution: If $X - X^\top = \mathbf{0}$, then we have $X = X^\top$, that is X is symmetric. Thus one choice of basis for null (T) is

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

(b) Find a basis for the range of T.

Solution: If $V = X - X^\top$, it is clear that

$$\begin{aligned} V^\top &= (X - X^\top)^\top \\ &= -X + X^\top \\ &= -(X - X^\top) \\ &= -V. \end{aligned}$$

Thus every vector in range (T) is skew symmetric, and a basis for range (T) is

$$\left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right).$$

(c) Verify the Fundamental Theorem for T.

Solution: We have $\dim(\text{null}\,(T)) = 6$, $\dim(\text{range}\,(T)) = 3$, and $\dim(\mathcal{M}_3(\mathbb{R})) = 9$. So clearly

$$\dim(\text{null}\,(T)) + \dim(\text{range}\,(T)) = \dim(\mathcal{M}_3(\mathbb{R})).$$

4. Find an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ so that $\text{null}\,(T) = \text{range}\,(T)$.

   Example: For any $x \in \mathbb{R}^4$, $T(x) \in \text{range}\,(T)$. Thus the stipulation $\text{null}\,(T) = \text{range}\,(T)$ implies that
   $$T(T(x)) = \mathbf{0}$$
   for all $x \in \mathbb{R}^4$. One possible way to build such an operator is
   $$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ x_1 \\ x_2 \end{pmatrix}.$$
   It is clear that T is indeed a linear transformation, and
   $$x \in \text{range}\,(T) \iff x = \begin{pmatrix} 0 \\ 0 \\ v \\ w \end{pmatrix} \iff T(x) = \mathbf{0}.$$

5. A linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ has matrix
   $$A = A_{(B,C)} = \begin{pmatrix} 1 & -3 & 0 \\ -4 & 13 & -1 \\ 8 & -25 & 2 \end{pmatrix}$$
   with respect to some bases B and C of $\mathcal{P}_2(\mathbb{R})$.

   (a) Is T injective? Explain why/why not.

   Solution: If T is injective, then $(T) = \{\mathbf{0}\}$, that is $T(v) = \mathbf{0} \iff v = \mathbf{0}$.
   Now $T(v) = \mathbf{0} \iff (T(v))_C = \mathbf{0}$, and since
   i. $A(v)_B = (T(v))_C$, and
   ii. every $x \in \mathbb{R}^3$ is the coordinate vector for some $p \in \mathcal{P}_2$,
   we may translate the observation into one on matrix equations: $Ax = \mathbf{0}$ has only the trivial solution if and only if $\det(A) \neq 0$.
   Now the determinant of A is $\det(A) = 1$, so the only solution to $A(v)_B = \mathbf{0}$ is the trivial one, $(v)_B = \mathbf{0}$. Thus T is indeed injective

   (b) Is T surjective? Explain why/why not.

   Solution: A linear operator is injective if and only if it is also surjective, so T is surjective.

6. Suppose that R, S, and T are linear operators on V so that RST is surjective. Prove that S is injective.

   Solution: Since RST is surjective, we know that for every $w \in V$, there is a $v \in V$ so that $RST(v) = w$. Since $RST(v) = R(S(T(v)))$, there is also a vector (namely $u := S(T(v))$) so that $R(u) = w$. Thus the operator R is surjective as well; this is equivalent to R injective. Thus for every $w \in V$, there is precisely one $v \in V$ so that $R(v) = w$.

Thus ST must be a surjective map, and using the same reasoning as above, S is surjective as well, and thus injective.

7. (*Deleted*) Recall Theorem 5.10:

   Let $T : V \to V$ be a linear operator on the finite dimensional vector space V. If $\lambda_1, \ldots, \lambda_n$ are *distinct* eigenvalues of T, and if $v_1, \ldots, v_n$ are vectors so that v_i is an eigenvector associated with λ_i, then the list $(v_1,\ v_2,\ \ldots,\ v_n)$ is an independent list.

   Fill in the details of the following sketch of the proof:

   Proceed by induction: show that, if v_1 and v_2 are eigenvectors for T and are also dependent, then they must be associated with the same eigenvalue.

   For the inductive hypothesis, let $v_1,\ \ldots,\ v_n$ be any eigenvectors associated with unique eigenvalues, so that $(v_1,\ \ldots,\ v_n)$ is an independent list. Let v_{n+1} be any eigenvector of T in $\operatorname{span}(v_1,\ \ldots,\ v_n)$, and show that v_{n+1} must be associated with one of the eigenvalues $\lambda_1, \ldots, \lambda_n$.