1. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by

$$T(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) = 2\alpha_1 x^3 + (\alpha_3 + \alpha_2)x + (\alpha_1 + \alpha_0).$$

(a) Is $x^3 - 5x^2 + 3x - 6$ in null $(T)$? Explain why/why not.
(b) Is $4x^3 - 4x^2$ in null $(T)$? Explain why/why not.
(c) Is $8x^3 - x - 1$ in range $(T)$? Explain why/why not.
(d) Is $4x^3 - 3x^2 + 7$ in range $(T)$? Explain why/why not.

2. Given

$$M = \begin{pmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{pmatrix},$$

define $T_M : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

(a) Find the rank of $M$.
(b) Find a basis for the null space of $T_M$.
(c) Find a basis for the range of $T_M$.
(d) Verify the Fundamental Theorem for $T_M$.

3. Define $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ by

$$T(X) = X - X^T.$$

(a) Find a basis for the null space of $T$.
(b) Find a basis for the range of $T$.
(c) Verify the Fundamental Theorem for $T$.

4. Find an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ so that null $(T) = \text{range} (T)$.

5. A linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ has matrix

$$A = A_{(B,C)} = \begin{pmatrix} 1 & -3 & 0 \\ -4 & 13 & -1 \\ 8 & -25 & 2 \end{pmatrix}$$

with respect to some bases $B$ and $C$ of $P_2(\mathbb{R})$.

(a) Is $T$ injective? Explain why/why not.
(b) Is $T$ surjective? Explain why/why not.

6. Suppose that $R$, $S$, and $T$ are linear operators on $V$ so that $RST$ is surjective. Prove that $S$ is injective.
7. Recall Theorem 5.10:

Let $T : V \to V$ be a linear operator on the finite dimensional vector space $V$. If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of $T$, and if $v_1, \ldots, v_n$ are vectors so that $v_i$ is an eigenvector associated with $\lambda_i$, then the list $(v_1, v_2, \ldots, v_n)$ is an independent list.

Fill in the details of the following sketch of the proof:

Proceed by induction: show that, if $v_1$ and $v_2$ are eigenvectors for $T$ and are also dependent, then they must be associated with the same eigenvalue.

For the inductive hypothesis, let $v_1, \ldots, v_n$ be any eigenvectors associated with unique eigenvalues, so that $(v_1, \ldots, v_n)$ is an independent list. Let $v_{n+1}$ be any eigenvector of $T$ in span $(v_1, \ldots, v_n)$, and show that $v_{n+1}$ must be associated with one of the eigenvalues $\lambda_1, \ldots, \lambda_n$. 