1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be given by

$$T(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) = 2\alpha_1 x^3 + (\alpha_3 + \alpha_2)x + (\alpha_1 + \alpha_0).$$

   (a) Is $x^3 - 5x^2 + 3x - 6$ in $\operatorname{null}(T)$? Explain why/why not.
   (b) Is $4x^3 - 4x^2$ in $\operatorname{null}(T)$? Explain why/why not.
   (c) Is $8x^3 - x - 1$ in $\operatorname{range}(T)$? Explain why/why not.
   (d) Is $4x^3 - 3x^2 + 7$ in $\operatorname{range}(T)$? Explain why/why not.

2. Given

$$M = \begin{pmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{pmatrix},$$

define $T_M : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

   (a) Find the rank of M.
   (b) Find a basis for the null space of T_M.
   (c) Find a basis for the range of T_M.
   (d) Verify the Fundamental Theorem for T_M.

3. Define $T : \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ by

$$T(X) = X - X^\top.$$

   (a) Find a basis for the null space of T.
   (b) Find a basis for the range of T.
   (c) Verify the Fundamental Theorem for T.

4. Find an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ so that $\operatorname{null}(T) = \operatorname{range}(T)$.

5. A linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ has matrix

$$A = A_{(B,C)} = \begin{pmatrix} 1 & -3 & 0 \\ -4 & 13 & -1 \\ 8 & -25 & 2 \end{pmatrix}$$

with respect to some bases B and C of $\mathcal{P}_2(\mathbb{R})$.

   (a) Is T injective? Explain why/why not.
   (b) Is T surjective? Explain why/why not.

6. Suppose that R, S, and T are linear operators on V so that RST is surjective. Prove that S is injective.

7. Recall Theorem 5.10:

   Let $T : V \to V$ be a linear operator on the finite dimensional vector space V. If $\lambda_1, \ldots, \lambda_n$ are *distinct* eigenvalues of T, and if $v_1, \ldots, v_n$ are vectors so that v_i is an eigenvector associated with λ_i, then the list $(v_1,\ v_2,\ \ldots,\ v_n)$ is an independent list.

   Fill in the details of the following sketch of the proof:

   Proceed by induction: show that, if v_1 and v_2 are eigenvectors for T and are also dependent, then they must be associated with the same eigenvalue.

   For the inductive hypothesis, let $v_1, \ldots, v_n$ be any eigenvectors associated with unique eigenvalues, so that $(v_1,\ \ldots,\ v_n)$ is an independent list. Let v_{n+1} be any eigenvector of T in span$(v_1,\ \ldots,\ v_n)$, and show that v_{n+1} must be associated with one of the eigenvalues $\lambda_1, \ldots, \lambda_n$.