1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^3$ be given by

$$T(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) = \begin{pmatrix} a_0 + a_1 \\ 2a_2 \\ a_3 - a_0 \end{pmatrix}.$$

   (a) Find the matrix A for T with respect to the standard bases $B = (x^3,\ x^2,\ x,\ 1)$ and $C = (e_1,\ e_2,\ e_3)$ for $\mathcal{P}_3(\mathbb{R})$ and $\mathbb{R}^3$ respectively.

   Solution:

$$A_{(B,B)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

   (b) Given $p(x) = -4x^2 - 3x - 5$, show that $A(p)_B = (T(p))_C$.

   Solution: On one hand, we have

$$\begin{aligned} A(p)_B &= \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \\ -3 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ -8 \\ 5 \end{pmatrix}. \end{aligned}$$

   On the other hand,

$$T(-4x^2 - 3x - 5) = \begin{pmatrix} -8 \\ -8 \\ 5 \end{pmatrix}.$$

   So clearly

$$A(p)_B = (T(p))_C.$$

2. Given the same transformation T above, but bases $D = (x^3 + x^2,\ x^2 + x,\ x + 1,\ 1)$ and

$$E = \left(\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} \right)$$

   for $\mathcal{P}_3(\mathbb{R})$ and $\mathbb{R}^3$ respectively, find the matrix for T.

Solution: Doing some arithmetic, we see that

$$\begin{aligned}
T(x^3+x^2) &= -\begin{pmatrix}-2\\1\\-3\end{pmatrix} + \begin{pmatrix}1\\-3\\0\end{pmatrix} - \begin{pmatrix}3\\-6\\2\end{pmatrix};\\
T(x^2+x) &= -10\begin{pmatrix}-2\\1\\-3\end{pmatrix} + 26\begin{pmatrix}1\\-3\\0\end{pmatrix} - 15\begin{pmatrix}3\\-6\\2\end{pmatrix};\\
T(x+1) &= -15\begin{pmatrix}-2\\1\\-3\end{pmatrix} + 41\begin{pmatrix}1\\-3\\0\end{pmatrix} - 23\begin{pmatrix}3\\-6\\2\end{pmatrix};\\
T(1) &= -9\begin{pmatrix}-2\\1\\-3\end{pmatrix} + 25\begin{pmatrix}1\\-3\\0\end{pmatrix} - 14\begin{pmatrix}3\\-6\\2\end{pmatrix}.
\end{aligned}$$

Thus the matrix $A = A_{(D,E)}$ for the transformation is given by

$$A = \begin{pmatrix}-1 & -10 & -15 & -9\\ 1 & 26 & 41 & 25\\ -1 & -15 & -23 & -14\end{pmatrix}.$$

3. Lists

$$B = \left(\begin{pmatrix}7 & 3\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}, \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} \right)$$

and

$$C = \left(\begin{pmatrix}22 & 7\\0 & 2\end{pmatrix}, \begin{pmatrix}12 & 4\\0 & 1\end{pmatrix}, \begin{pmatrix}33 & 12\\0 & 2\end{pmatrix} \right)$$

are both bases for $\mathcal{U}_2(\mathbb{R})$.

(a) Find the transition matrix X from B to C.

Solution: Doing a bit of arithmetic to write the coordinates for vectors in B in terms of vectors in C, we see that

$$\begin{pmatrix}7 & 3\\0 & 0\end{pmatrix} = \begin{pmatrix}22 & 7\\0 & 2\end{pmatrix} - 4\begin{pmatrix}12 & 4\\0 & 1\end{pmatrix} + \begin{pmatrix}33 & 12\\0 & 2\end{pmatrix};$$

$$\begin{pmatrix}1 & 2\\0 & -1\end{pmatrix} = -2\begin{pmatrix}22 & 7\\0 & 2\end{pmatrix} + \begin{pmatrix}12 & 4\\0 & 1\end{pmatrix} + \begin{pmatrix}33 & 12\\0 & 2\end{pmatrix};$$

and

$$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}22 & 7\\0 & 2\end{pmatrix} + \begin{pmatrix}12 & 4\\0 & 1\end{pmatrix} - \begin{pmatrix}33 & 12\\0 & 2\end{pmatrix}.$$

Thus the transition matrix X from B to C is given by

$$X = \begin{pmatrix}1 & -2 & 1\\ -4 & 1 & 1\\ 1 & 1 & -1\end{pmatrix}.$$

(b) Vector $v \in \mathcal{U}_2(\mathbb{R})$ has coordinates

$$(v)_B = \begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix}.$$

Find v.

Solution: Clearly

$$\begin{aligned} v &= 4\begin{pmatrix} 7 & 3 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} - 6\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 24 \\ 0 & -9 \end{pmatrix}. \end{aligned}$$

(c) Use your transition matrix to find $(v)_C$.

Solution:

$$\begin{aligned} (v)_C &= X(v)_B \\ &= \begin{pmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ -19 \\ 13 \end{pmatrix}. \end{aligned}$$

4. Let $T_\theta : \mathbb{R}^2 \to R^2$ be the linear transformation that rotates a vector $x \in \mathbb{R}^2$ by angle θ counterclockwise.

(a) Show that

$$T_\theta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}.$$

Solution: First suppose $v \in V$ is a unit vector, with

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};$$

if v makes angle θ' with the positive part of the x axis, then $x_1 = \cos\theta_1$ and $x_2 = \sin\theta_1$. Let

$$v' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

be the rotation of v counterclockwise by θ; then v' makes an angle of $\theta' + \theta$ with the positive part of the x axis, so that

$$\begin{aligned} x_1' &= \cos(\theta' + \theta) \\ &= \cos\theta' \cos\theta - \sin\theta' \sin\theta \\ &= x_1 \cos\theta - x_2 \sin\theta \end{aligned}$$

and

$$\begin{aligned} x_2' &= \sin(\theta' + \theta) \\ &= \sin\theta' \cos\theta + \cos\theta' \sin\theta \\ &= x_1 \sin\theta + x_2 \cos\theta. \end{aligned}$$

Thus

$$T_\theta(v) = \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}.$$

The argument still works if v is not a unit vector, since T is a linear transformation.

(b) Find the matrix A for T_θ with respect to the standard basis $B = (e_1,\ e_2)$ for $\mathbb{R}^2$.

Solution: Since

$$T_\theta(e_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ and } T_\theta(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

we have

$$A_{(B,B)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

5. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix

$$A = \begin{pmatrix} -3 & 7 \\ 0 & 4 \end{pmatrix}$$

with respect to the standard basis $B = (e_1,\ e_2)$ for $\mathbb{R}^2$. Find a basis B' for $\mathbb{R}^2$ so that the matrix for T with respect to B' is diagonal.

Example: If X is an invertible 2×2 matrix with

$$XAX^{-1} = A',$$

then A' can be thought of as the matrix for T with respect to *some* basis B' of V. One possible way to tackle this problem is to find an X so that XAX^{-1} is diagonal, and then "read off" the change of basis from matrix X.

We can assume that $\det(X) = 1$, so that

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then

$$XAX^{-1} = \begin{pmatrix} (-7a+4b)c - 3ad & 3ab + a(7a+4b) \\ -c(7d+7c) & 3bc + a(7c+4d) \end{pmatrix}.$$

We want to guarantee that $A' = XAX^{-1}$ is diagonal; one way to do this is to let $c = 0$, so that

$$XAX^{-1} = \begin{pmatrix} -3ad & 3ab + a(7a+4b) \\ 0 & 4ad \end{pmatrix}.$$

Now

$$\det(X) = ad = 1,$$

so the equation again simplifies to

$$XAX^{-1} = \begin{pmatrix} -3 & 3ab + a(7a+4b) \\ 0 & 4 \end{pmatrix}.$$

We wish to choose a and b so that A^{-1} is diagonal, that is we want

$$3ab + a(7a+4b) = 0.$$

We can easily accomplish this by setting $a = 1$ and $b = -1$, so that we have

$$X = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } XAX^{-1} = A' = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}.$$

Now X^{-1} is the transition matrix from basis $B' = (v_1,\ v_2)$ to basis B; thus we may read off the coordinates of the basis vectors in B' from the columns of

$$X^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}:$$

the first column indicates that $v_1 = e_1$, and the second column indicates that $v_2 = e_1 + e_2$; thus one possible choice for basis B' is

$$B' = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$$

6. Let V be an n dimensional vector space over $\mathbb{F}$, and let $B = (v_1,\ v_2,\ \ldots,\ v_n)$ be a basis for V. Show that $(v_i)_B = e_i$ for all i, $1 \le i \le n$, where $e_i \in \mathbb{F}^n$ is the $n \times 1$ matrix with 1 in the ith entry and 0s elsewhere.

   Solution: Since the decomposition

   $$v_i = 0v_1 + 0v_2 + \ldots + 0v_{i-1} + 1v_i + 0v_{i+1} + \ldots + 0v_n$$

   is unique, the coordinate vector for v_i with respect to B must have 0s in each entry except for a 1 in the ith entry. Thus

   $$(v)_B = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_i.$$

7. Let V be a finite dimensional vector space over $\mathbb{F}$ with bases B and C. Prove that, if $X \in \mathbb{M}_n(\mathbb{F})$ is the transition matrix from B to C, then X is invertible, and X^{-1} is the transition matrix from C to B.

   Solution: Let X be the transition matrix from B to C, so that

   $$X(v)_B = (v)_C$$

   for each $v \in V$. Let

   $$B = (v_1,\ v_2,\ \ldots,\ v_n) \text{ and } C = (w_1,\ w_2,\ \ldots,\ w_n).$$

   It is clear that each $y \in \mathbb{F}^n$ is the coordinate vector of at least one vector in V. So the equation

   $$Xy = \mathbf{0}$$

   has only the trivial solution; indeed, for $y = (v)_B$, $Xy = (v)_C$, we have

   $$Xy = \mathbf{0} \iff v = 0w_1 + 0w_2 + \ldots + 0w_n = \mathbf{0}.$$

   Thus X is invertible.

   On one hand

   $$X^{-1}X(v)_B = (v)_B,$$

   while on the other hand

   $$X^{-1}(X(v)_B) = X^{-1}(v)_C.$$

   So for each $v \in V$,

   $$X^{-1}(v)_C = (v)_B$$

   and X^{-1} is the transition matrix from C to B.