1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^3$ be given by

$$T(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) = \begin{pmatrix} a_0 + a_1 \\ 2a_2 \\ a_3 - a_0 \end{pmatrix}.$$

   (a) Find the matrix A for T with respect to the standard bases $B = (x^3,\ x^2,\ x,\ 1)$ and $C = (e_1,\ e_2,\ e_3)$ for $\mathcal{P}_3(\mathbb{R})$ and $\mathbb{R}^3$ respectively.

   (b) Given $p(x) = -4x^2 - 3x - 5$, show that $A(p)_B = (T(p))_C$.

2. Given the same transformation T above, but bases $D = (x^3 + x^2,\ x^2 + x,\ x + 1,\ 1)$ and

$$E = \left(\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} \right)$$

   for $\mathcal{P}_3(\mathbb{R})$ and $\mathbb{R}^3$ respectively, find the matrix for T.

3. Lists

$$B = \left(\begin{pmatrix} 7 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)$$

   and

$$C = \left(\begin{pmatrix} 22 & 7 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 12 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 33 & 12 \\ 0 & 2 \end{pmatrix} \right)$$

   are both bases for $\mathcal{U}_2(\mathbb{R})$.

   (a) Find the transition matrix X from B to C.

   (b) Vector $v \in \mathcal{U}_2(\mathbb{R})$ has coordinates

$$(v)_B = \begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix}.$$

   Find v.

   (c) Use your transition matrix to find $(v)_C$.

4. Let $T_\theta : \mathbb{R}^2 \to R^2$ be the linear transformation that rotates a vector $x \in \mathbb{R}^2$ by angle θ counterclockwise.

   (a) Show that

$$T_\theta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}.$$

   (b) Find the matrix A for T_θ with respect to the standard basis $B = (e_1,\ e_2)$ for $\mathbb{R}^2$.*

5. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix

$$A = \begin{pmatrix} -3 & 7 \\ 0 & 4 \end{pmatrix}$$

   with respect to the standard basis $B = (e_1,\ e_2)$ for $\mathbb{R}^2$. Find a basis B' for $\mathbb{R}^2$ so that the matrix for T with respect to B' is diagonal.

6. Let V be an n dimensional vector space over $\mathbb{F}$, and let $B = (v_1,\ v_2,\ \ldots,\ v_n)$ be a basis for V. Show that $(v_i)_B = e_i$ for all i, $1 \leq i \leq n$, where $e_i \in \mathbb{F}^n$ is the $n \times 1$ matrix with 1 in the ith entry and 0s elsewhere.

7. Let V be a finite dimensional vector space over $\mathbb{F}$ with bases B and C. Prove that, if $X \in \mathbb{M}_n(\mathbb{F})$ is the transition matrix from B to C, then X is invertible, and X^{-1} is the transition matrix from C to B.

*Notice that you've seen this matrix before–A is actually *orthogonal*, that is $A^\top = A^{-1}$.