1. Recall that the set $\mathbb{R}_+$ of all positive real numbers is a vector space with the following definitions for addition and scalar multiplication (to avoid confusion, we use the symbol $\Box$ to refer to the addition operation, and $\wedge$ to refer to the operation of scalar multiplication):

$$a \Box b = ab \text{ and } \lambda \wedge a = a^{\lambda}.$$

(a) Find a basis B for $\mathbb{R}^+$, and justify your answer.
Example: One possible choice is $B = (2)$, since for any $x \in \mathbb{R}^+$, $\log_2(x) \in \mathbb{R}$ so that

$$\begin{aligned} x &= 2^{\log_2(x)} \\ &= (\log_2 x) \wedge 2. \end{aligned}$$

(b) Define the map $T : \mathbb{R}^+ \to \mathbb{R}$ by $T(u) = \ln u$. Evaluate $T(e \Box e^4)$ and $T(-3 \wedge 5)$. (Notice that we have *not* proved that T is a linear transformation. Therefore you should not assume the properties of a linear transformation when making the calculations above).
Solution: Since $e \Box e^4 = e * e^4 = e^5$, we have

$$\begin{aligned} T(e \Box e^4) &= T(e^5) \\ &= \ln(e^5) \\ &= 5. \end{aligned}$$

Similarly, $-3 \wedge 5 = 5^{-3}$, so

$$\begin{aligned} T(-3 \wedge 5) &= T(5^{-3}) \\ &= \ln(5^{-3}) \\ &= -3 \ln 5. \end{aligned}$$

(c) Given the basis B you chose in part (a), define a map $T' : B \to \mathbb{R}$ by

$$T'(b) = \ln b \ \forall b \in B.$$

Describe the action of T' on each vector in B.
Solution: The only basis vector was 2, so we have

$$T'(2) = \ln 2.$$

(d) Recall that T' defines a unique linear transformation (which we will call T' as well) on all of $\mathbb{R}^+$. Show that T is a linear transformation by proving that $T = T'$. (Hint: Use the change of base formula for logarithms).
Solution: Since every vector in $\mathbb{R}^+$ can be written in the form $\alpha \wedge 2$ for some $\alpha \in \mathbb{R}$, we extend T' to a map on $\mathbb{R}^+$ as follows:

$$\begin{aligned} T'(\alpha \wedge 2) &= \alpha T'(2) \\ &= \alpha \ln 2. \end{aligned}$$

Now we want to show that T and T' match up. Given $x \in \mathbb{R}^+$ so that $x = \alpha \wedge 2$, we know that $\alpha = \log_2(x)$. So we have

$$\begin{aligned} T'(x) &= T'((\log_2(x)) \wedge 2) \\ &= \log_2(x) \cdot \ln 2 \\ &= \left(\frac{\ln x}{\ln 2}\right) \ln 2 \\ &= \ln x \\ &= T(x). \end{aligned}$$

2. Show that T from the problem above is a linear transformation using the definition of transformation.

   Solution: Per usual, we need to show that $T(x \square y) = T(x) + T(y)$ and $T(\lambda \wedge x) = \lambda T(x)$:

$$\begin{aligned} T(x \square y) &= T(xy) \\ &= \ln(xy) \\ &= \ln x + \ln y \\ &= T(x) + T(y). \end{aligned}$$

   Similarly,

$$\begin{aligned} T(\lambda \wedge x) &= T(x^{\lambda}) \\ &= \ln x^{\lambda} \\ &= \lambda \ln x \\ &= \lambda T(x). \end{aligned}$$

3. Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and let U be a subspace of V. Let $T : U \to W$ be a linear transformation. Show that T can be extended to a linear transformation $T' : V \to W$ (that is, T' is a linear transformation so that $T'(u) = T(u)$ whenever $u \in U$). (Not required, but interesting to think about: Is T' unique?)

   Solution: Let $S = (u_1, \ldots, u_n)$ be a basis for U, which we may extend to a basis $\hat{S} = (u_1, \ldots, u_n, v_1, \ldots, v_m)$ for V. Now set $T'(u) = T(u)$ for all $u \in S$, and choose arbitrary $w_i \in W$, $1 \leq i \leq m$, and set $T'(v_i) = w_i$. Now since we have defined T' on a basis for V, T' extends uniquely to a linear transformation $T' : V \to W$, and clearly $T'(u) = T(u)$ for all $u \in U$ (since T' is also a linear transformation on U).

   Once we have chosen the images w_i of the v_i, T' is unique. However, there are infinitely many ways to choose these images, so there are infinitely many different linear transformations that we can build from T.

4. Let $X \in \mathcal{M}_n(\mathbb{F})$ be a matrix with $\det(X) \neq 0$ so that X^{-1} exists. Show that the map

$$c_X : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$$

defined by

$$c_X(A) = XAX^{-1}$$

is a linear transformation.

Solution: Given A, $B \in \mathcal{M}_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} c_X(A+B) &= X(A+B)X^{-1} \\ &= XAX^{-1} + XBX^{-1} \\ &= c_X(A) + c_X(B), \end{aligned}$$

and

$$\begin{aligned} c_X(\lambda A) &= X(\lambda A)X^{-1} \\ &= \lambda XAX^{-1} \\ &= \lambda c_X(A). \end{aligned}$$

5. Given

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

consider the linear transformation $c_X : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ defined in the previous problem.

(a) Find the matrix A for c_X with respect to the standard basis $B = (e_{11},\ e_{12},\ e_{21},\ e_{22})$ for $\mathcal{M}_2(\mathbb{R})$.

Solution: We need to describe the action of X on each of the basis vectors. Setting

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we have

$$\begin{aligned}
c_X(e_1) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= e_4; \\
c_X(e_2) &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\
&= -e_3; \\
c_X(e_3) &= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
&= -e_2; \\
c_X(e_4) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= e_1.
\end{aligned}$$

With respect to the standard basis, we have coordinates

$$\begin{aligned}
(c_X(e_1)) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
(c_X(e_2)) &= \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \\
(c_X(e_3)) &= \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \\
(c_X(e_5)) &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus the matrix A for c_X with respect to the standard basis is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Show that, for any $Y \in \mathcal{M}_{\in}(\mathbb{R})$, $A(Y)_B = (c_X(Y))_B$.

Solution: On one hand, given

$$Y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with coordinates

$$(Y)_B = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix},$$

we have

$$\begin{aligned} A(Y)_B &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \\ &= \begin{pmatrix} \delta \\ -\gamma \\ -\beta \\ \alpha \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} c_X(Y) &= XYX^{-1} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma & -\delta \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}. \end{aligned}$$

Thus the coordinates of $c_X(Y)$ are given by

$$(c_X(Y))_B = \begin{pmatrix} \delta \\ -\gamma \\ -\beta \\ \alpha \end{pmatrix} = A(Y)_B.$$