1. Recall that the set $\mathbb{R}_+$ of all positive real numbers is a vector space with the following definitions for addition and scalar multiplication (to avoid confusion, we use the symbol $\Box$ to refer to the addition operation, and $\wedge$ to refer to the operation of scalar multiplication):

$$a \Box b = ab \text{ and } \lambda \wedge a = a^{\lambda}.$$

   (a) Find a basis B for $\mathbb{R}^+$, and justify your answer.

   (b) Define the map $T : \mathbb{R}^+ \to \mathbb{R}$ by $T(u) = \ln u$. Evaluate $T(e \Box e^4)$ and $T(-3 \wedge 5)$. (Notice that we have *not* proved that T is a linear transformation. Therefore you should not assume the properties of a linear transformation when making the calculations above).

   (c) Given the basis B you chose in part (a), define a map $T' : B \to \mathbb{R}$ by

$$T'(b) = \ln b \ \forall b \in B.$$

   Describe the action of T' on each vector in B.

   (d) Recall that T' defines a unique linear transformation (which we will call T' as well) on all of $\mathbb{R}^+$. Show that T is a linear transformation by proving that $T = T'$ (Hint: Use the change of base formula for logarithms).

2. Show that T from the problem above is a linear transformation using the definition of transformation.

3. Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and let U be a subspace of V. Let $T : U \to W$ be a linear transformation. Show that T can be extended to a linear transformation $T' : V \to W$ (that is, T' is a linear transformation so that $T'(u) = T(u)$ whenever $u \in U$). (Not required, but interesting to think about: Is T' unique?)

4. Let $X \in \mathcal{M}_n(\mathbb{F})$ be a matrix with $\det(X) \neq 0$ so that X^{-1} exists. Show that the map

$$c_X : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$$

   defined by

$$c_X(A) = XAX^{-1}$$

   is a linear transformation.

5. Given

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

   consider the linear transformation $c_X : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ defined in the previous problem.

   (a) Find the matrix A for c_X with respect to the standard basis $B = (e_{11},\ e_{12},\ e_{21},\ e_{22})$ for $\mathcal{M}_2(\mathbb{R})$.

   (b) Show that, for any $Y \in \mathcal{M}_{\in}(\mathbb{R})$, $A(Y)_B = (c_X(Y))_B$.