1. Recall that the vector space $\mathfrak{sl}(2,\mathbb{R})$ of $2\times 2$ trace 0 matrices is a subspace of $\mathcal{M}_2(\mathbb{R})$. The list

$$\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right).$$

Find *two different* subspaces $W_1$ and $W_2$ of $\mathcal{M}_2(\mathbb{R})$ so that

$$\mathcal{M}_2(\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \oplus W_1 \text{ and } \mathfrak{sl}(2,\mathbb{R}) \oplus W_2.$$

*Example*: Since $\mathfrak{sl}(2,\mathbb{R})$ is 3 dimensional, $W$ should be 1 dimensional. Two possible choices are

$$W_1 = \text{span}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)$$

and

$$W_2 = \text{span}\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right).$$

2. Let $V$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\left(\begin{pmatrix}11\\4\\1\\10\end{pmatrix}, \begin{pmatrix}2\\1\\1\\1\end{pmatrix}, \begin{pmatrix}8\\1\\-5\\13\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\-2\end{pmatrix}\right).$$

   (a) Determine the dimension of $V$.
   *Solution*: $V$ is 2 dimensional, as is shown below.

   (b) Find a basis for $V$.
   *Example*: The span of the list above is a vector space, so if the list itself is dependent, then we may reduce it to a basis for $V$ by deleting dependent vectors. Thus we look for nonzero solutions to

$$\alpha\begin{pmatrix}11\\4\\1\\10\end{pmatrix} + \beta\begin{pmatrix}2\\1\\1\\1\end{pmatrix} + \gamma\begin{pmatrix}8\\1\\-5\\13\end{pmatrix} + \delta\begin{pmatrix}-1\\0\\1\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

   Per usual, we look at the coefficient matrix

$$\begin{pmatrix}11 & 2 & 8 & -1\\ 4 & 1 & 1 & 0\\ 1 & 1 & -5 & 1\\ 10 & 1 & 13 & -2\end{pmatrix}$$

for the corresponding matrix equation, and row reduce to find solutions:

$$\begin{pmatrix} 11 & 2 & 8 & -1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & -5 & 1 \\ 10 & 1 & 13 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -5 & 1 \\ 11 & 2 & 8 & -1 \\ 4 & 1 & 1 & 0 \\ 10 & 1 & 13 & -2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & -5 & 1 \\ 0 & -9 & 63 & -12 \\ 0 & -3 & 21 & -4 \\ 0 & -9 & 63 & -12 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & -5 & 1 \\ 0 & 1 & -7 & 4/3 \\ 0 & -3 & 21 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & -5 & 1 \\ 0 & 1 & -7 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 2 & -1/3 \\ 0 & 1 & -7 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\gamma$ and $\delta$ are free variables, and we parametrize them accordingly, say $\gamma = 1$, $\delta = 3$, so that $\alpha = -1$ and $\beta = 3$. Thus

$$-\begin{pmatrix} 11 \\ 4 \\ 1 \\ 10 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and we rewrite

$$\begin{pmatrix} 11 \\ 4 \\ 1 \\ 10 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}.$$

Thus

$$\operatorname{span}\left(\begin{pmatrix} 11 \\ 4 \\ 1 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}\right);$$

if these three vectors are independent, then they form a basis for $V$. We check using the same process:

$$\begin{pmatrix} 2 & 8 & -1 \\ 1 & 1 & 0 \\ 1 & -5 & 1 \\ 1 & 13 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -1/2 \\ 0 & -3 & 1/2 \\ 0 & -9 & 3/2 \\ 0 & 9 & -3/2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 4 & -1/2 \\ 0 & 1 & -1/6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1/6 \\ 0 & 1 & -1/6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, we have a free variable: parameterizing $c = 6$, so that $b = 1$ and $a = -1$, we have

$$-\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix} + 6\begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

again rewriting as

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix} + 6\begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}.$$

So

$$\operatorname{span}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}\right);$$

these last two vectors are clearly independent, so the list

$$\left(\begin{pmatrix} 8 \\ 1 \\ -5 \\ 13 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}\right)$$

is a basis for $V$.

3. Let

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & -1 & 5 \\ 1 & 0 & 1 \end{pmatrix}.$$

We can think of the columns of $A$ as vectors in $\mathbb{R}^3$. The subspace of $\mathbb{R}^3$ spanned by the columns of $A$ is called the *column space* of $A$, denoted by column $(A)$.

(a) Find two nonzero vectors in column $(A)$.

*Example*: We could, of course, just pick two columns of $A$, but more interesting possibilities are

$$\begin{pmatrix}6\\8\\2\end{pmatrix} = 2\begin{pmatrix}1\\2\\1\end{pmatrix} - 4\begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

and

$$\begin{pmatrix}4\\-4\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\\0\end{pmatrix} - \begin{pmatrix}4\\5\\1\end{pmatrix}.$$

(b) Show that column $(A) \neq \mathbb{R}^3$ using an argument on the determinant of $A$.

*Solution*: Since $\det(A) = 0$, there are nontrivial solutions to the matrix equation

$$\begin{pmatrix}1 & -1 & 4\\2 & -1 & 5\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

i.e. there are nonzero scalars such that

$$x_1\begin{pmatrix}1\\2\\1\end{pmatrix} + x_2\begin{pmatrix}-1\\-1\\0\end{pmatrix} + x_3\begin{pmatrix}4\\5\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Thus the three vectors are *not* independent, so that column $(A) \neq \mathbb{R}^3$.

(c) Find a basis for column $(A)$ and determine the dimension of column $(A)$.

*Example*: Since the list of vectors

$$\left(\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}, \begin{pmatrix}4\\5\\1\end{pmatrix}\right)$$

is not independent, but *does* span column $(A)$, it contains a basis for column $(A)$; deleting one (or two) carefully chosen vectors will result in the desired basis.

In essence, we would like to find a nonzero solution to the matrix equation

$$\begin{pmatrix}1 & -1 & 4\\2 & -1 & 5\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

so let's begin by row-reducing the coefficient matrix:

$$\begin{pmatrix} 1 & -1 & 4 \\ 2 & -1 & 5 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $x_3$ is free; choosing $x_3 = 1$, we have $x_2 = 3$ and $x_1 = -1$. Thus

$$-\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

is a linear combination of the other two vectors, we may safely remove it from the list. Thus

$$\left( \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \right)$$

still spans column $(A)$, and we can quickly row-reduce the resulting coefficient matrix to see that the vectors above are linearly independent:

$$\begin{pmatrix} -1 & 4 \\ 1 & 5 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -4 \\ 1 & 5 \\ 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -4 \\ 0 & 9 \\ 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so that $a = b = 0$ is the only solution to

$$\begin{pmatrix} -1 & 4 \\ 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$\left( \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \right)$$

is a basis for column $(A)$, and $\dim(\text{column}\,(A)) = 2$.

4. Let $V$ be the subspace of $\mathcal{P}_4(\mathbb{R})$ of all vectors $p \in \mathcal{P}_4(\mathbb{R})$ so that

$$p''(1/2) = 0.$$

(a) Find a basis for $V$.

*Example*: If $p \in V$ with

$$p(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \lambda,$$

then

$$p''(x) = 12\alpha x^2 + 6\beta x + 2\gamma,$$

so that

$$\begin{aligned} 0 &= p''(1/2) \\ &= 3\alpha + 3\beta + 2\gamma. \end{aligned}$$

We have

$$\gamma = -\frac{3}{2}\alpha - \frac{3}{2}\beta,$$

so that

$$p(x) = \alpha x^4 + \beta x^3 + (-\frac{3}{2}\alpha - \frac{3}{2}\beta)x^2 + \delta x + \lambda.$$

Thus every vector in $V$ may be written in the form

$$p(x) = \alpha(x^4 - \frac{3}{2}x^2) + \beta(x^3 - \frac{3}{2}x^2) + \delta x + \lambda,$$

so that the vectors

$$x^4 - \frac{3}{2}x^2,\ x^3 - \frac{3}{2}x^2,\ x,\ 1$$

form a spanning list for $V$. It is straightforward to show that this list is also independent, so the list

$$(x^4 - \frac{3}{2}x^2,\ x^3 - \frac{3}{2}x^2,\ x,\ 1)$$

is also a basis for $V$.

(b) Extend the basis to a basis for $\mathcal{P}_4(\mathbb{R})$.

*Example*: $V$ itself is 4 dimensional, and $\mathcal{P}_4(\mathbb{R})$ is 5 dimensional, so we need to add a single vector to the basis to extend it to a basis for all of $\mathcal{P}_4(\mathbb{R})$. I claim that we can add $x^2$ and maintain independence of the list; to be certain, we check the coefficient matrix for the system

$$\alpha(x^4 - \frac{3}{2}x^2) + \beta(x^3 - \frac{3}{2}x^2) + \delta x + \lambda + \mu x^2 = \mathbf{0}.$$

The matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -3/2 & -3/2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which has determinant 1. Since the determinant is nonzero, the only solution to the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -3/2 & -3/2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is the trivial solution

$$\begin{pmatrix} \alpha \\ \beta \\ \delta \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the vectors are independent, and the list

$$(x^4 - \frac{3}{2}x^2,\ x^3 - \frac{3}{2}x^2,\ x,\ 1,\ x^2)$$

is a basis for $\mathcal{P}_4(\mathbb{R})$.

5. Prove that every subspace of $\mathbb{R}^3$ is either $\{\mathbf{0}\}$, a line through the origin, or a plane through the origin.

*Solution*: Every subspace $V$ of $\mathbb{R}^3$ must have dimension

$$\dim(V) \leq 3.$$

If $\dim V = 3$, then clearly $V = \mathbb{R}^3$.

We handle the remaining cases seperately:

(a) $\dim V = 0$: Then the empty list is a basis for $V$, and $\operatorname{span}() = \{\mathbf{0}\}$.

(b) $\dim V = 1$: Then there is a single vector $u \neq \mathbf{0}$ in $\mathbb{R}^3$ so that

$$V = \operatorname{span}(u),$$

say

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Then every vector $v \in V$ may be written in the form

$$v = \alpha \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{pmatrix},$$

$\alpha \in \mathbb{R}$. Now the set of all scalar multiples of a nonzero vector in $\mathbb{R}^3$ is a line, and particular this line must pass through the origin ($\alpha = 0$).

(c) $\dim V = 2$: There is a pair of (nonzero) vectors $u$, $v \in \mathbb{R}^3$ so that

$$V = \operatorname{span}(u,\ v),$$

say

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Then every vector $w \in V$ may be written in the form

$$w = \alpha \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \beta \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$\alpha$, $\beta \in \mathbb{R}$. But the set of all vectors of this form is just a plane in $\mathbb{R}^3$, again passing through the origin since we can set $\alpha = \beta = 0$.

6. Prove that if $U$ and $W$ are both 5 dimensional subspaces of $\mathbb{R}^9$, then $U \cap W \neq \{\mathbf{0}\}$.

*Solution*:

Recall that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

If $U \cap W = \{\mathbf{0}\}$, then $\dim(U \cap W) = 0$, and we have

$$\dim(U + W) = \dim U + \dim W = 10.$$

However, $U + W$ is a subspace of the 9 dimensional vector space $V$; we now have a contradiction, since the dimension of a subspace of $V$ cannot be more than the dimension of $V$ itself. Thus $U \cap W \neq \{\mathbf{0}\}$.

7. The notation

$$\dim_{\mathbb{F}}(V)$$

denotes the dimension of $V$ as a vector space over field $\mathbb{F}$.

Let $V$ be any vector space over $\mathbb{C}$. Since $\mathbb{R} \subset \mathbb{C}$, $V$ is also a vector space over $\mathbb{R}$. Prove that

$$\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V).$$

*Solution*: Suppose that $\dim_{\mathbb{C}}(V) = n$; that is, every basis of $V$ over $\mathbb{C}$ contains $n$ vectors. Let

$$B_{\mathbb{C}} = (v_1,\ v_2,\ \ldots,\ v_n)$$

be one such basis.

I claim that

$$B_{\mathbb{R}} = (v_1,\ v_2,\ \ldots,\ v_n,\ iv_1,\ iv_2,\ \ldots,\ iv_n)$$

is a basis for $V$ over $\mathbb{R}$. To prove the claim, we need to show that $B_{\mathbb{R}}$ is independent and spanning.

To show that $B_{\mathbb{R}}$ spans $V$ over $\mathbb{R}$, suppose that $v \in V$. Now $B_{\mathbb{C}}$ spans $V$ over $\mathbb{C}$, so we know that there constants $\alpha_j \in \mathbb{C}$ so that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = v.$$

Of course, each $\alpha_j$ may be rewritten as

$$\alpha_j = a_j + ib_j,$$

$a_j$, $b_j \in \mathbb{R}$, so that we may rewrite $v$ as a linear combination of vectors in $B_{\mathbb{R}}$ as follows:

$$\begin{aligned} v &= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \\ &= (a_1 + b_1 i)v_1 + (a_2 + b_2 i)v_2 + \ldots + (a_n + b_n i)v_n \\ &= a_1 v_1 + a_2 v_2 + \ldots + a_n v_n + b_1(iv_1) + b_2(iv_2) + \ldots + b_n(iv_n); \end{aligned}$$

since all of the scalars above are elements of $\mathbb{R}$, this is the desired linear combination of vectors in $B_{\mathbb{R}}$, which clearly spans $V$ over $\mathbb{R}$.

To show that the vectors are independent over $\mathbb{R}$, recall that the only constants $\alpha_i \in \mathbb{C}$ so that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \mathbf{0}$$

are

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

Thus if

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n + b_1(iv_1) + b_2(iv_2) + \ldots + b_n(iv_n) = \mathbf{0},$$

we may rewrite this as a linear combination of vectors in $B_{\mathbb{C}}$ by setting $\beta_j = a_j + ib_j$. Then we have

$$\begin{aligned} \mathbf{0} &= a_1 v_1 + a_2 v_2 + \ldots + a_n v_n + b_1(iv_1) + b_2(iv_2) + \ldots + b_n(iv_n) \\ &= \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n, \end{aligned}$$

so that $\beta_j = 0$ for all $j$ by the above argument. Thus $a_j = b_j = 0$ for all $j$, and the vectors in $B_{\mathbb{R}}$ are independent over $\mathbb{R}$.