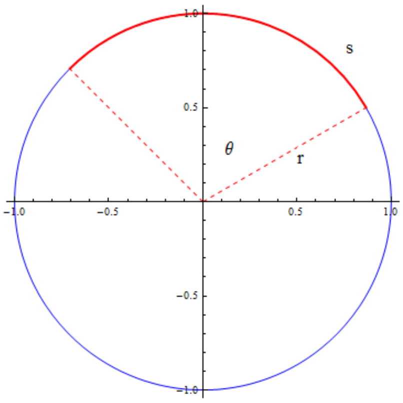
Given an arc of length s on a circle of radius r, the radian measure of the central angle subtended by the arc is given by $\theta = \frac{s}{r}$:

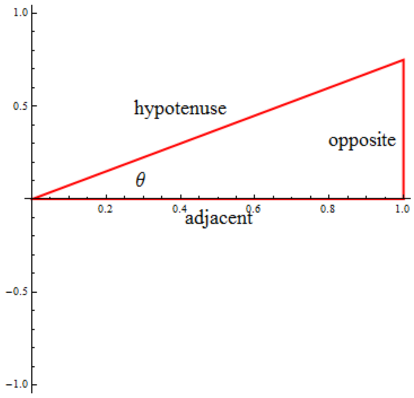


To convert from radians (rad) to degrees ($\circ$) and vice versa, use the following conversions:

$$1rad = \frac{180^\circ}{\pi}, \qquad 1^\circ = \frac{\pi}{180}rad.$$

In particular, $180^\circ = \pi rad$.

Given a right triangle (a triangle one of whose angles is $\frac{\pi}{2}rad$), choose one of the acute angles ($< \frac{\pi}{2}rad$), and call it θ. We label the sides relative to θ as follows:



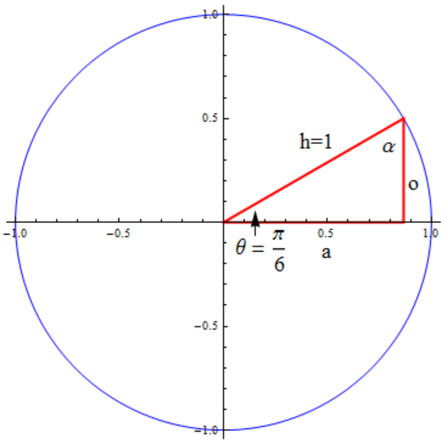
Let o, a, and h be the lengths of the opposite side, adjacent side, and hypoteneuse, respectively. We use these numbers to define the following functions:

$$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h} \qquad \tan\theta = \frac{o}{a} = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{h}{o} = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{h}{a} = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{a}{o} = \frac{1}{\tan\theta}$$

Some "special" angles have particularly nice trigonometric properties; we can use a unit circle (a circle of radius 1) to determine the trigonometric values for such angles.

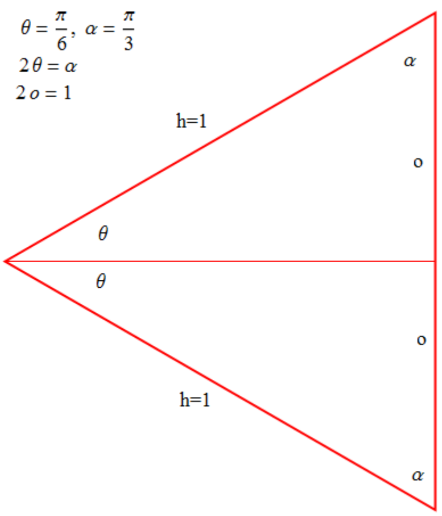
First, let's consider the special angle $\theta = \frac{\pi}{6}$:



We can determine the third angle in the right triangle above, since the sum of the degrees in the angles of any triangle must be πrad. The third angle is

$$\alpha = \pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}.$$

Now that we know values for all of its angles, let's inspect the triangle above more closely; we would like to find values for $\sin(\frac{\pi}{6})$, $\cos(\frac{\pi}{6})$, etc. To do so, we must determine the lengths of each of the sides; fortunately, we know that the length of the hypoteneuse is 1 since the triangle was embedded in a unit circle. Let's "double" the triangle, as depicted below:



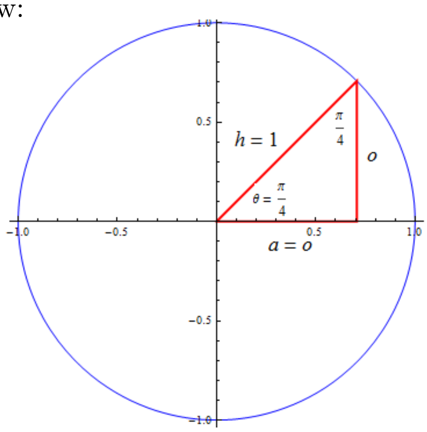
This new larger triangle is equiangular (all of its angles are $\frac{\pi}{3}$), thus is also equilateral–all of its sides have length 1. It is clear that the length of the opposite side is $o = \frac{1}{2}$, and we can use the

Pythagorean identity $o^2 + a^2 = h^2$ to see that $a = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$.

So we have

$$\sin\frac{\pi}{6} = \frac{o}{h} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{a}{h} = \frac{\sqrt{3}}{2}, \quad \text{and} \quad \tan\frac{\pi}{6} = \frac{o}{a} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Let's do the same thing for $\theta = \frac{\pi}{4}$. The unit circle and right triangle for this case are graphed below:



It is clear that the remaining angle has measure $\frac{\pi}{4}$, so that $a = o$. By the Pythagorean identity, $a^2 + o^2 = h^2$, which we rewrite as $a^2 + a^2 = 1$ or $2a^2 = 1$. So $a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$. Then

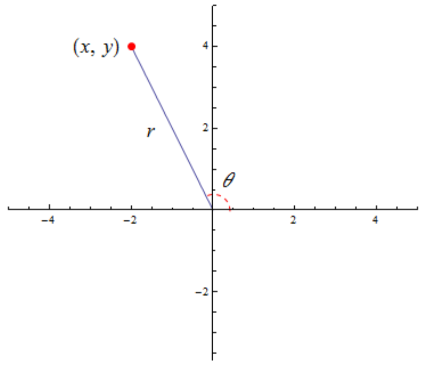
$$\sin\frac{\pi}{4} = \frac{o}{h} = \frac{1}{\sqrt{2}}, \quad \cos\frac{\pi}{4} = \frac{a}{h} = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \tan\frac{\pi}{4} = \frac{o}{a} = 1.$$

Below is a table of the values of the sine, cosine, and tangent functions at special angles in the first quadrant:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Our current definitions for the six trigonometric functions only apply to acute angles, i.e. angles less than $\frac{\pi}{2}$; however, we may extend the definitions to apply to *any* angle.

Given an angle θ in the xy plane, we may draw a ray from the origin in the appropriate direction, pick a point (x, y) on the ray, and label the segment from the origin to (x, y) with r:

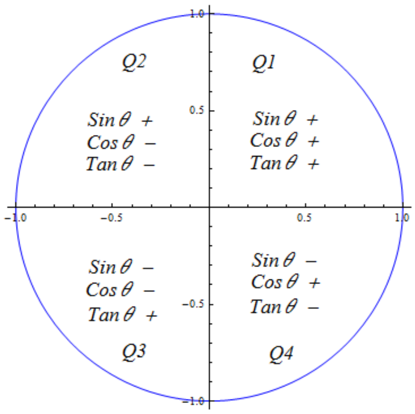


Then we define the trigonometric functions for angle θ as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$
$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

If $\theta < \frac{\pi}{2}$, then these definitions are precisely the same as the earlier definitions; these newer definitions simply allow us to apply the trigonometric functions to angles greater than $\frac{\pi}{2}$.

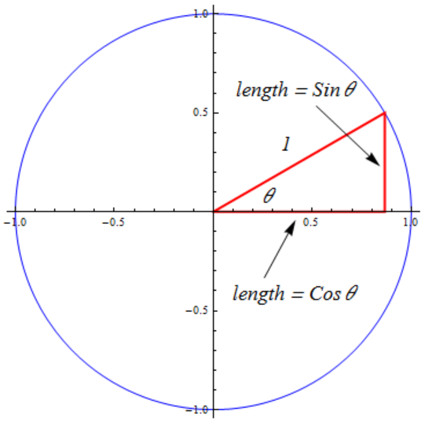
Note that the definitions tell us the signs of each of the trig functions in the different quadrants:

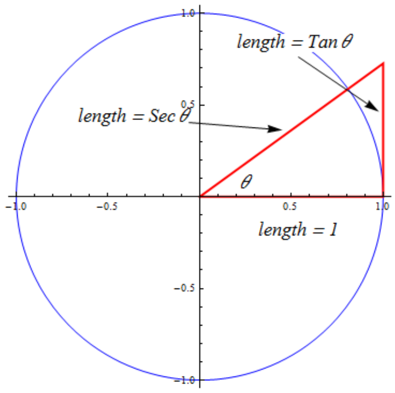


Below is a list of the domains and ranges of the various trigonometric functions:

Function	Domain	Range
$\sin\theta$	all real numbers	$[-1, 1]$
$\cos\theta$	all real numbers	$[-1, 1]$
$\tan\theta$	all real numbers except $\frac{\pi}{2}$, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, $-\frac{3\pi}{2}$, etc.	all real numbers
$\cot\theta$	all real numbers except 0, π, $-\pi$, 2π, etc.	all real numbers
$\sec\theta$	all real numbers except $\frac{\pi}{2}$, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, $-\frac{3\pi}{2}$, etc.	$(-\infty, -1] \cup [1, \infty)$
$\csc\theta$	all real numbers except 0, π, $-\pi$, 2π, etc.	$(-\infty, -1] \cup [1, \infty)$

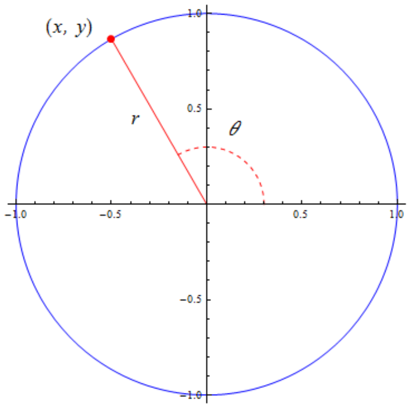
Another helpful set of facts to have at our disposal involves the definitions of the trig functions on the unit circle. Since $h = 1$, it is easy to check the accuracy of the following diagrams:



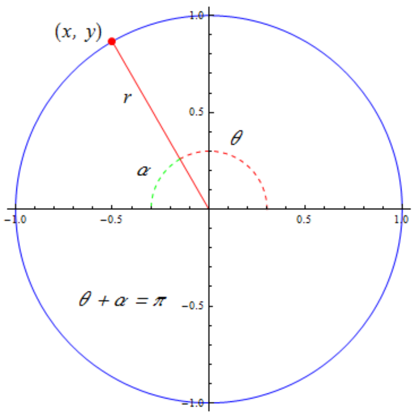


Finding the values of trig functions for values of θ that do not lie in the first quadrant is made much simpler by using reference angles, which allow us to return to acute angles.

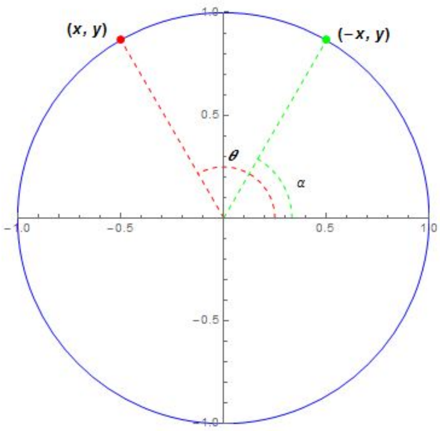
For example, consider finding $\sin\theta$ in the unit circle ($r = 1$) below:



In this example on the unit circle with $r = 1$, $\sin\theta = \frac{y}{r} = y$. Now consider the angle α in the graph below:

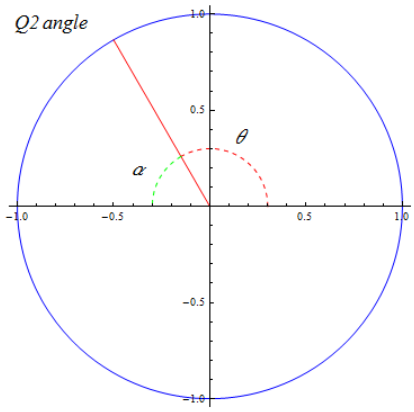


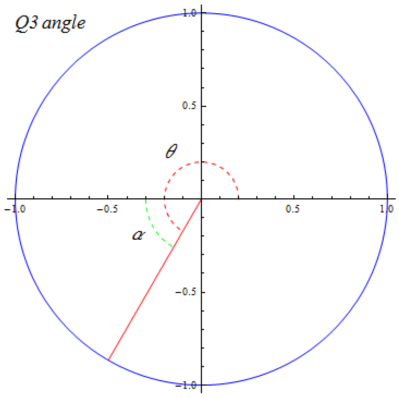
We can think of this acute angle as an angle in the first quadrant:

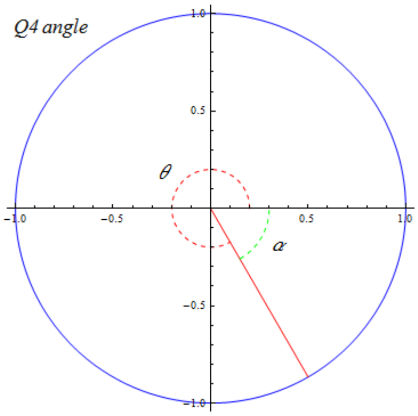


Notice that the endpoints of the line segments above have the same height or y coordinate. Because of this, $\sin\alpha = y$ as well; in fact, if $\theta + \alpha = \pi$, then $\sin\theta = \sin\alpha$ for any $\frac{\pi}{2} < \theta\pi$. Since it is easier to evaluate trig functions on acute angles, we would really prefer to work with α, and we call α a *reference angle* for θ.

We can actually use a similar process for each of the trig functions in each of the four quadrants; the reference angle α in each quadrant is graphed below:

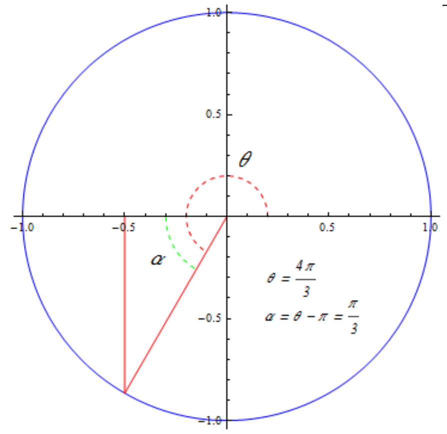




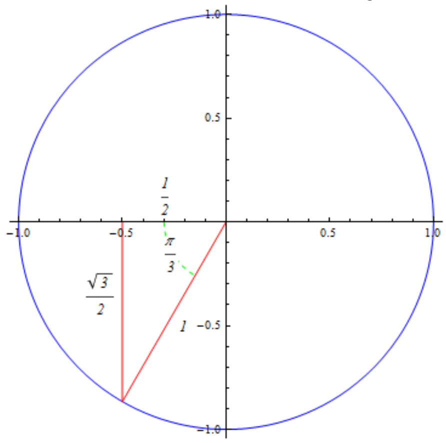


To determine the value for $\sin\theta$, $\cos\theta$, or $\tan\theta$, simply evaluate the trig function on the reference angle α, then change the sign of the answer according to whether the function is positive or negative on the quadrant in which θ lies.

For example, let's find $\sin\frac{4\pi}{3}$, $\cos\frac{4\pi}{3}$, and $\tan\frac{4\pi}{3}$:



The reference angle for $\theta = \frac{4\pi}{3}$ is $\alpha = \frac{\pi}{3}$.



In addition, the sine and cosine functions are negative in the third quadrant, whereas the tangent function is positive. Since

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \text{ and } \tan\frac{\pi}{3} = \sqrt{3},$$

we see that

$$\sin\frac{4\pi}{3} =, -\frac{\sqrt{3}}{2}, \quad \cos\frac{4\pi}{3} = -\frac{1}{2}, \text{ and } \tan\frac{4\pi}{3} = \sqrt{3}.$$

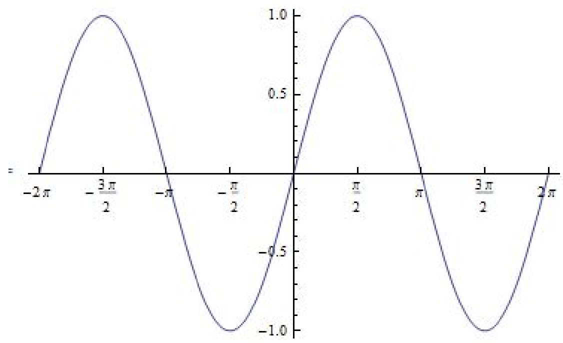
Finally, here are a few important identities to keep in mind when working with trig functions:

$$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = \tan^2\theta + 1, \quad \csc^2\theta = \cot^2\theta + 1$$

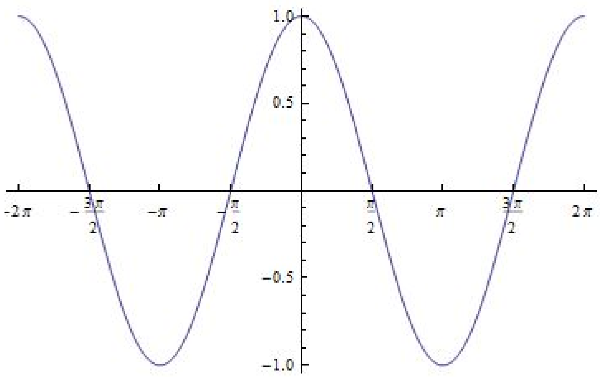
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin^2\alpha = \frac{1-\cos(2\alpha)}{2}.$$

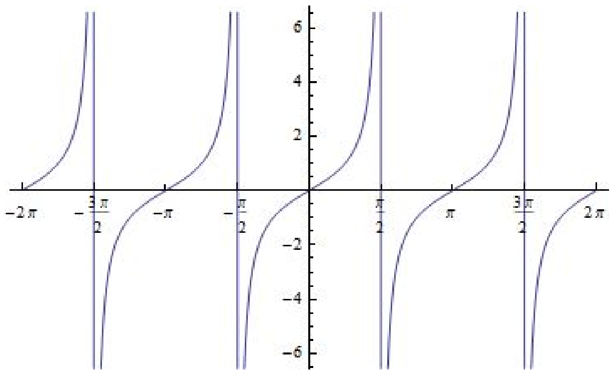
Below are graphs of the various trig functions.



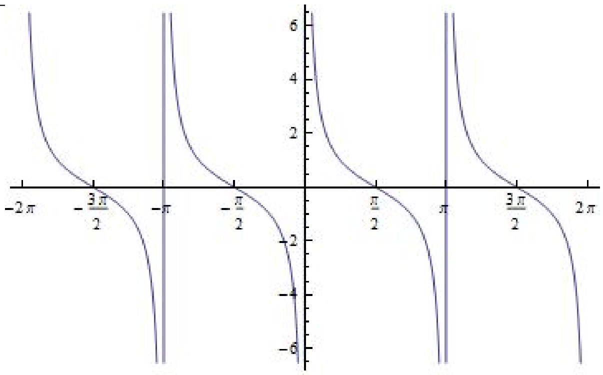
$\sin\theta$



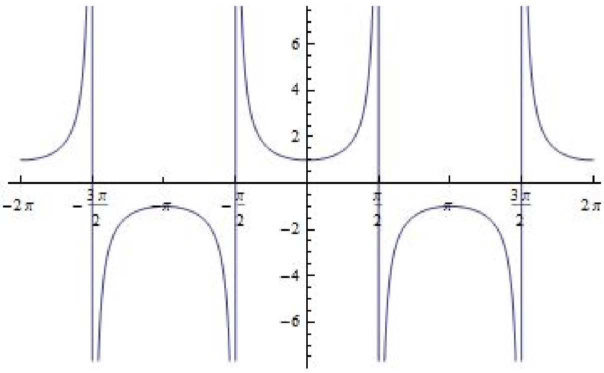
$\cos\theta$



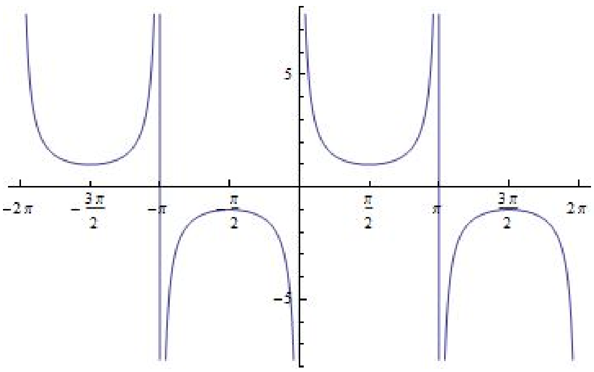
$\tan\theta$



$\cot\theta$



$\sec\theta$



$\csc\theta$