Geometry of Inner Product Spaces

In Calculus III, you learned to calculate the *angle* θ between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ using the dot product via the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\right).$$

Since the dot product is an example of an inner product, it seems reasonable to guess that we can define "angles" between vectors in a general inner product space by replacing the dot product above with the inner product in the ambient space, say as

$$\theta = \cos^{-1}\left(\frac{\langle\mathbf{u},\mathbf{v}\rangle}{||\mathbf{u}||\ ||\mathbf{v}||}\right).$$

Before we define the angle between a pair of vectors in a general inner product space, however, we must do a bit of bookkeeping.

The Cauchy-Schwarz Inequality

Since the definition of angle involves the arccosine function $\cos^{-1} x$, whose domain is $[-1, 1]$, we will eventually need to make sure that the input for the function is indeed a real number between -1 and 1. The Cauchy-Schwarz inequality will allow us to do just that:

Theorem 1.6.2. *Cauchy-Schwarz Inequality:* If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a (real) inner product space V equipped with the inner product $\langle\cdot,\cdot\rangle$, then

$$|\langle\mathbf{u},\mathbf{v}\rangle| \le ||\mathbf{u}||\ ||\mathbf{v}||.$$

In particular, the theorem states that the (absolute value of the) inner product of a pair of vectors will never be more than the product of their lengths (measured under the same inner product).

If neither of $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector, so that $||\mathbf{u}|| > 0$ and $||\mathbf{v}|| > 0$, then we may rewrite the inequality as

$$\frac{|\langle\mathbf{u},\mathbf{v}\rangle|}{||\mathbf{u}||\ ||\mathbf{v}||} \le 1 \text{ or } -1 \le \frac{\langle\mathbf{u},\mathbf{v}\rangle}{||\mathbf{u}||\ ||\mathbf{v}||} \le 1.$$

The last inequality above is particularly nice, because we now know that *any number of the form*

$$\frac{\langle\mathbf{u},\mathbf{v}\rangle}{||\mathbf{u}||\ ||\mathbf{v}||}$$

can be an input for the arccosine function.

Angle Between Vectors in an Inner Product Space

With this last observation in mind, we now define the angle between vectors in a general inner product space:

Definition. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in a (real) inner product space V equipped with the inner product $\langle \cdot, \cdot \rangle$, then the *angle* θ between vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||\ ||\mathbf{v}||} \right).$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if the angle θ between them is given by

$$\theta = \frac{\pi}{2},$$

or equivalently if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

Remark. It is important to recall that the range of the arccosine function is *defined* to be $[0, \pi]$.

Example

Use the standard inner product on P_2 to find the angle between vectors

$$\mathbf{f} = f(x) = 1 - x \text{ and } \mathbf{g} = g(x) = x^2.$$

Since the angle θ between the vectors is given by

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{f}, \mathbf{g} \rangle}{||\mathbf{f}||\ ||\mathbf{g}||} \right),$$

we need to calculate $\langle \mathbf{f}, \mathbf{g} \rangle$, $||\mathbf{f}||$, and $||\mathbf{g}||$.

The standard inner product on P_3 multiplies the coefficients of corresponding terms and then sums the results. So we have

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle &= 1 \cdot 0 + (-1 \cdot 0) + (0 \cdot 1) \\ &= 0. \end{aligned}$$

Let's calculate the lengths of each of $\mathbf{f}$ and $\mathbf{g}$:

$$\begin{aligned} ||\mathbf{f}|| &= \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2}, \end{aligned}$$

and

$$\begin{aligned} ||\mathbf{g}|| &= \sqrt{\langle \mathbf{g}, \mathbf{g} \rangle} \\ &= \sqrt{1^2} \\ &= 1. \end{aligned}$$

Thus the angle θ between $\mathbf{f}$ and $\mathbf{g}$ is given by

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\langle \mathbf{f}, \mathbf{g} \rangle}{||\mathbf{f}||\ ||\mathbf{g}||}\right) \\ &= \cos^{-1}\left(\frac{0}{\sqrt{2} \cdot 1}\right) \\ &= \cos^{-1} 0 \\ &= \frac{\pi}{2}. \end{aligned}$$

Note that, under the standard inner product on P_2, these two vectors are orthogonal.

Example

Find the angle between vectors $\mathbf{f}$ and $\mathbf{g}$ above under the alternate inner product defined on P_2 by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(x)g(x)\ \mathrm{d}x.$$

With $f(x) = 1 - x$ and $g(x) = x^2$, we need to find $\langle \mathbf{f}, \mathbf{g} \rangle$, $||\mathbf{f}||$, and $||\mathbf{g}||$ under this new inner product:

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle &= \int_{-1}^{1} (1-x)x^2\ \mathrm{d}x \\ &= \int_{-1}^{1} x^2 - x^3\ \mathrm{d}x \\ &= \frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_{-1}^{1}\ \mathrm{d}x \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} + \frac{1}{4}\Big|_{-1}^{1}\ \mathrm{d}x \\ &= \frac{2}{3}. \end{aligned}$$

Next, we calculate the lengths of each of the vectors:

$$\begin{aligned}
||\mathbf{f}|| &= \sqrt{\int_{-1}^{1} (f(x))^2 \ \mathrm{d}x} \\
&= \sqrt{\int_{-1}^{1} (1-x)^2 \ \mathrm{d}x} \\
&= \sqrt{\int_{-1}^{1} 1 - 2x + x^2 \ \mathrm{d}x} \\
&= \sqrt{x - x^2 + \frac{1}{3}x^3 \Big|_{-1}^{1}} \\
&= \sqrt{1 - 1 + \frac{1}{3} + 1 + 1 + \frac{1}{3}} \\
&= \sqrt{2 + \frac{2}{3}} \\
&= \sqrt{\frac{8}{3}},
\end{aligned}$$

while

$$\begin{aligned}
||\mathbf{g}|| &= \sqrt{\int_{-1}^{1} (g(x))^2 \ \mathrm{d}x} \\
&= \sqrt{\int_{-1}^{1} (x^2)^2 \ \mathrm{d}x} \\
&= \sqrt{\int_{-1}^{1} x^4 \ \mathrm{d}x} \\
&= \sqrt{\frac{1}{5}x^5 \Big|_{-1}^{1}} \\
&= \sqrt{\frac{1}{5} + \frac{1}{5}} \\
&= \sqrt{\frac{2}{5}}.
\end{aligned}$$

Thus under this new inner product on P_2, the angle θ between $\mathbf{f}$ and $\mathbf{g}$ is given by

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\langle \mathbf{f}, \mathbf{g}\rangle}{||\mathbf{f}||\ ||\mathbf{g}||}\right) \\
&= \cos^{-1}\left(\frac{\frac{2}{3}}{\sqrt{\frac{8}{3}\cdot\frac{2}{5}}}\right) \\
&= \cos^{-1}\left(\frac{2}{3\sqrt{\frac{8}{3}\cdot\frac{2}{5}}}\right) \\
&= \cos^{-1}\left(\frac{2}{3\sqrt{\frac{16}{15}}}\right) \\
&= \cos^{-1}\left(\frac{2}{4\sqrt{\frac{9}{15}}}\right) \\
&= \cos^{-1}\left(\frac{1}{2\sqrt{\frac{3}{5}}}\right) \\
&= \cos^{-1}\left(\frac{\sqrt{5}}{2\sqrt{3}}\right) \\
&\approx .87 \text{ radians} \\
&= 49.8^\circ.
\end{aligned}$$

Notice that, under this particular choice of an inner product, vectors $\mathbf{f}$ and $\mathbf{g}$ are *not* orthogonal.

Key Point. The lengths of vectors and distances and angles between a pair of vectors are quantities that depend completely on the choice of inner product. Under the first choice of inner product above, vectors $\mathbf{f}$ and $\mathbf{g}$ were orthogonal; under the second choice, they were not.

Example

Find the angle between vectors

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$$

using the inner product on $\mathcal{M}_2$ defined by

$$\langle U, V\rangle = \operatorname{tr}(U^\top V).$$

Again, we need to use the formula

$$\theta = \cos^{-1}\left(\frac{\langle U, V\rangle}{||U||\ ||V||}\right);$$

in the last section, we calculated that

$$||U|| = 1 \text{ and } ||V|| = 3\sqrt{2}.$$

Thus we simply need to calculate

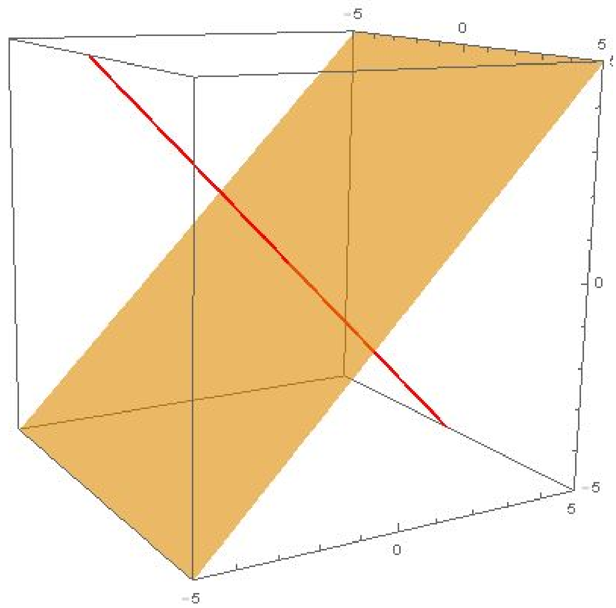
$$\begin{aligned}
\langle U, V\rangle &= U^\top V \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^\top \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}\right) \\
&= \operatorname{tr}\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{pmatrix} \\
&= \frac{5}{2} - \frac{1}{2} \\
&= 2.
\end{aligned}$$

Thus the angle θ between U and V is given by

$$\begin{aligned}
\theta = \cos^{-1}\left(\frac{\langle U, V\rangle}{||U||\ ||V||}\right) \\
&= \cos^{-1}\left(\frac{2}{1 \cdot 3\sqrt{2}}\right) \\
&= \cos^{-1}\left(\frac{2}{3\sqrt{2}}\right) \\
&\approx 1.08 \text{ radians} \\
&= 61.87^\circ.
\end{aligned}$$

Orthogonal Complements

Now that we understand what it means for a pair of vectors in a general inner product space to be orthogonal, we can revisit the idea of an *orthogonal complement* which we introduced for $\mathbb{R}^n$ in Section 4.8. The orthogonal complement $W^\perp$ of a subspace W of $\mathbb{R}^n$ is the set of all vectors from $\mathbb{R}^n$ that are orthogonal to *every* vector in W. For example, the line and the plane graphed below in $\mathbb{R}^3$ are orthogonal complements:



Of course, the idea of orthogonal complements now makes sense for *any* inner product space. Accordingly, we define them in more generality than we did in Section 4.8:

Definition 2. If W is a subspace of a real inner product space V with inner product given by $\langle \cdot, \cdot \rangle$, then the *orthogonal complement* $W^\perp$ of W is the set of all vectors in V that are orthogonal to every vector in W under $\langle \cdot, \cdot \rangle$.

Just as in $\mathbb{R}^n$, orthogonal complements of subspaces are also subspaces:

Theorem 6.2.2. If W is a subspace of a real inner product space V, then $W^\perp$ is also a subspace of V.

Example

Find a basis for the orthogonal complement of the subspace of P_2 consisting of all scalar multiples of $\mathbf{f} = f(x) = x^2$ under the standard inner product on P_2.

Recall that the standard inner product of vectors

$$\mathbf{f} = f(x) = a_0 + a_1 x + a_2 x^2 \text{ and } \mathbf{g} = g(x) = b_0 + b_1 x + b_2 x^2$$

in P_2 is given by

$$\langle \mathbf{f}, \mathbf{g} \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2.$$

We need to find a basis for the subspace of P_2 consisting of all functions $\mathbf{g} = g(x) = b_0 + b_1 x + b_2 x^2$ in P_2 so that

$$\langle \mathbf{f}, \mathbf{g} \rangle = 0.$$

Accordingly, we calculate

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle &= \langle x^2, b_0 + b_1 x + b_2 x^2 \rangle \\ &= 0 \cdot b_0 + 0 \cdot b_1 + 1 \cdot b_2 \\ &= b_2. \end{aligned}$$

Since we want to find the vectors $\mathbf{g}$ so that $\langle \mathbf{f}, \mathbf{g} \rangle = 0$, we must choose $b_2 = 0$. However b_0 and b_1 are free, so $W^\perp$ consists of vectors of the form $\mathbf{g} = g(x) = b_0 + b_1 x$.

To find a basis for $W^\perp$, we simply need to note that we can think of $W^\perp$ as the set of all linear combinations of 1 and x; thus the desired basis is

$$\{1, x\}.$$