Rank and Nullity

In section 4.7, we defined the *row space* and *column space* of a matrix A as the vector spaces spanned by the rows and columns of A, respectively. For example, we saw that the row space of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 4 & 0 \\ 2 & 6 & 2 & -10 & 0 \\ 3 & 9 & 2 & -18 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

is the three dimensional vector space spanned by the vectors

$$\begin{aligned} \mathbf{r_1}' &= \begin{pmatrix} 1 & 3 & 1 & -5 & 0 \end{pmatrix} \\ \mathbf{r_2}' &= \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\ \mathbf{r_3}' &= \begin{pmatrix} 0 & 0 & 1 & 3 & -1 \end{pmatrix}; \end{aligned}$$

the column space of A is the three dimensional vector space spanned by the vectors

$$\mathbf{c_1} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \ \mathbf{c_2} = \begin{pmatrix} 1 \\ 6 \\ 9 \\ 1 \end{pmatrix}, \text{ and } \mathbf{c_3} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

It is interesting to note that, while the column space and row space of matrix A are not the same vector space (indeed the row space is "living" in $\mathbb{R}^5$, whereas the column space is in $\mathbb{R}^4$), they *are* vector spaces of the same dimension. We will see in this section that this is no fluke. We will explore this idea and many more of the interconnections among row space, column space, null space, and solutions to the system $A\mathbf{x} = \mathbf{b}$ in this section.

Rank and Nullity

Let's think about why our matrix A above had the row space and column space of the same dimension: we reduced A by Gaussian elimination to

$$R = \begin{pmatrix} 1 & 3 & 1 & -5 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

R and A have the same row space, and while they *do not* have the same column space, their column spaces have the same dimension.

So we can ask our question in a different way: why do the dimensions of the column space and row space of R match up? Let's inspect R again:

$$R = \begin{pmatrix} 1 & 3 & 1 & -5 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We saw a theorem in 4.7 that told us how to find the row space and column space for a matrix in row echelon form:

Theorem. If a matrix R is in row echelon form, then the row vectors with leading 1s form a basis for the row space of R (and for any matrix row equivalent to R), and the column vectors with leading 1s form a basis for the column space of R.

In other words, the dimensions of the column spaces and row spaces are determined by the number of leading 1s in columns and rows, respectively. The leading 1s for R are highlighted in red below:

$$R = \begin{pmatrix} 1 & 3 & 1 & -5 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

notice the the leading 1s of the rows are also the leading 1s of the columns. That is, *every* leading 1 is a leading 1 for both a row and a column; in general, any matrix in row echelon form has the same number of leading 1s in its rows as it does in its columns, thus its row space and column space must have the same dimension. We make this idea precise in the next theorem:

Theorem 4.8.1. The row space and column space of a matrix A have the same dimension.

We name the shared dimensions of the row and column spaces of A, as well as the dimension of the vector space $\text{null}\,(A)$, in the following:

Definition 1. The dimension of the row space/column space of a matrix A is called the *rank* of A; we use notation $\text{rank}\,(A)$ to indicate that

$$\dim(\text{row}\,(A)) = \dim(\text{column}\,(A)) = \text{rank}\,(A).$$

The dimension of the vector space $\text{null}\,(A)$ is called the *nullity* of A, and is denoted $\text{nullity}\,(A)$.

Example

Given

$$A = \begin{pmatrix} 0 & 1 & 1 & 4 & 0 \\ 2 & 6 & 2 & -10 & 0 \\ 3 & 9 & 2 & -18 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

find:

1. $\text{rank}\,(A)$
2. $\text{nullity}\,(A)$

We investigated matrix A in Section 4.7; to find $\text{rank}\,(A)$, we simply need to determine the dimension of either $\text{row}\,(A)$ or $\text{column}\,(A)$ (they're the same number!), and to find $\text{nullity}\,(A)$, we need to know the dimension of $\text{null}\,(A)$.

1. We saw that the vector space row (A) has basis

$$\begin{aligned}
\mathbf{r_1}' &= \begin{pmatrix}1 & 3 & 1 & -5 & 0\end{pmatrix} \\
\mathbf{r_2}' &= \begin{pmatrix}0 & 1 & 0 & 1 & 1\end{pmatrix} \\
\mathbf{r_3}' &= \begin{pmatrix}0 & 0 & 1 & 3 & -1\end{pmatrix},
\end{aligned}$$

   so its dimension is 3. We conclude that

$$\operatorname{rank}(A) = 3.$$

2. The null space null (A) has basis

$$\begin{pmatrix}11\\-1\\-3\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}2\\-1\\1\\0\\1\end{pmatrix}.$$

   Thus the dimension of null $(A) = 2$, and we see that

$$\operatorname{nullity}(A) = 2.$$

Example

Given

$$A = \begin{pmatrix}1 & 3 & 1 & 2 & 5\\0 & 1 & 1 & 0 & -1\\0 & 0 & 0 & 1 & 6\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{pmatrix},$$

find

1. rank (A)
2. nullity (A).

Since A is already in row echelon form, the questions should be relatively easy to answer.

1. The rank of A is the dimension of the row space (and column space) of A. This corresponds to the number of leading 1s in rows (or columns) of A; the leading 1s are highlighted in red below:

$$A = \begin{pmatrix}\color{red}{1} & 3 & 1 & 2 & 5\\0 & \color{red}{1} & 1 & 0 & -1\\0 & 0 & 0 & \color{red}{1} & 6\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

Notice that the 1 in position $2,3$ is *not* a leading 1 of a column, as the column before it (column 2) has a leading 1 in the same position. Since there are three leading 1s, we know that the vector spaces $\text{row}\,(A)$ and $\text{column}\,(A)$ both have dimension 3, so that

$$\text{rank}\,(A) = 3.$$

2. Let's calculate the dimension of the null space of A, that is the dimension of the solution space to

$$A\mathbf{x} = \mathbf{0}.$$

The augmented equation for the system is

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 2 & 5 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right);$$

the third row tells us that

$$x_4 = -6x_5,$$

and the second row says that

$$x_2 = -x_3 + x_5.$$

Combining these equalities with the data from the first row, we have

$$\begin{aligned} x_1 &= -3x_2 - x_3 - 2x_4 - 5x_5 \\ &= -3(-x_3 + x_5) - x_3 - 2(-6x_5) - 5x_5 \\ &= 3x_3 - 3x_5 - x_3 + 12x_5 - 5x_5 \\ &= 2x_3 + 4x_5. \end{aligned}$$

Thus any vector $\mathbf{x}$ in the solution space to the equation $R\mathbf{x} = \mathbf{0}$ has form

$$\mathbf{x} = \begin{pmatrix} 2x_3 + 4x_5 \\ -x_3 + x_5 \\ x_3 \\ -6x_5 \\ x_5 \end{pmatrix};$$

so the solution space consists of all linear combinations of the form

$$x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix}.$$

The null space of A is thus two-dimensional, so that

$$\text{nullity}\,(A) = 2.$$

Dimension Theorems

Just as with the example we investigated in Section 4.7, we see that the row space of A above is a three-dimensional subspace of $\mathbb{R}^5$; since $\text{row}\,(A)$ took up three dimensions of $\mathbb{R}^5$, there were only two dimensions left for $\text{null}\,(A)$. We make these ideas more precise in the following theorem.

Theorem 4.8.2. If A is an $m \times n$ matrix (in particular, A has n columns) then

$$\text{rank}\,(A) + \text{nullity}\,(A) = n.$$

If A is $m \times n$, then the row space and null space of A are both subspaces of $\mathbb{R}^n$. As indicated in the previous examples, the theorem states that the row space and null space "use up" all of $\mathbb{R}^n$.

Key Point. Recall that the column space (a subspace of $\mathbb{R}^m$) and the row space (a subspace of $\mathbb{R}^n$) must have the same dimension. In this case, the maximum value for $\dim(\text{column}\,(A))$ is m, and the maximum value for $\dim(\text{row}\,(A))$ is n. So

$$\dim(\text{row}\,(A)) = \dim(\text{column}\,(A)) = \min(m, n).$$

Example

A matrix A is 4×9. Find:

1. The maximum possible value for $\text{rank}\,(A)$ and the minimum possible value for $\text{nullity}\,(A)$.
2. $\text{rank}\,(A)$ given that $\text{nullity}\,(A) = 7$.

1. Since A has 4 rows and 9 columns, the maximum possible value for $\text{rank}\,(A)$ is 4, and we know that
$$\text{rank}\,(A) + \text{nullity}\,(A) = 9.$$
Thus $\text{nullity}\,(A)$ must be at least 5, and will be more if $\text{rank}\,(A) < 4$.
2. If $\text{nullity}\,(A) = 7$, then $\text{rank}\,(A) = 2$ since
$$\text{rank}\,(A) = 9 - \text{nullity}\,(A).$$

Rank and Nullity of A and $A^\top$

Recall that the matrix $A^\top$ is obtained from A by interchanging rows and columns, as in

$$A = \begin{pmatrix} 3 & 9 & 3 & 6 & 15 \\ -1 & -2 & 0 & -2 & -6 \\ 0 & 2 & 2 & 1 & 4 \\ 0 & -1 & -1 & -\frac{1}{2} & -2 \\ 2 & 9 & 5 & 5 & 13 \\ 3 & 12 & 6 & \frac{15}{2} & 21 \end{pmatrix}, \; A^\top = \begin{pmatrix} 3 & -1 & 0 & 0 & 2 & 3 \\ 9 & -2 & 2 & -1 & 9 & 12 \\ 3 & 0 & 2 & -1 & 5 & 6 \\ 6 & -2 & 1 & -\frac{1}{2} & 5 & \frac{15}{2} \\ 15 & -6 & 4 & -2 & 13 & 21 \end{pmatrix}.$$

Because of the simple relationship between A and $A^\top$, we can very quickly determine some information about column space, row space, and rank of $A^\top$ if we already know the corresponding data about A.

First of all, since the *rows* of A are the *columns* of $A^\top$, and vice versa, it is clear that

$$\operatorname{row}(A) = \operatorname{column}(A^\top),$$

and that

$$\operatorname{column}(A) = \operatorname{row}(A^\top).$$

It is important to note, however, that in general

$$\operatorname{null}(A) \neq \operatorname{null}(A^\top)$$

Of course, if $\operatorname{row}(A) = \operatorname{column}(A^\top)$, then these vector spaces clearly have the same dimension:

$$\dim(\operatorname{row}(A)) = \dim(\operatorname{column}(A^\top)).$$

In addition, we know that

$$\operatorname{rank}(A) = \dim(\operatorname{row}(A)) = \dim(\operatorname{column}(A)),$$

and that

$$\operatorname{rank}(A^\top) = \dim(\operatorname{row}(A^\top)) = \dim(\operatorname{column}(A^\top));$$

combining these observations, we see that

$$\operatorname{rank}(A) = \operatorname{rank}(A^\top).$$

This is actually a proof of the following theorem:

Theorem 4.8.5. For any matrix A, A and $A^\top$ have the same rank, that is

$$\operatorname{rank}(A) = \operatorname{rank}(A^\top).$$

Given all of the data that we have seen about the interconnections between A and $A^\top$, we should now pinpoint four vector spaces that are closely related to the pair A and $A^\top$:

Definition. The four *fundamental spaces* of matrices A and $A^\top$ are

$$\begin{array}{cc} \text{row}\,(A) = \text{column}\,(A^\top) & \text{column}\,(A) = \text{row}\,(A^\top) \\ \text{null}\,(A) & \text{null}\,(A^\top). \end{array}$$

We can make a few more quick observations about these spaces:

Key Point. If A is $m \times n$, so that $A^\top$ is $n \times m$, then we know that

$$\text{rank}\,(A) + \text{nullity}\,(A) = n$$

and

$$\text{rank}\,(A^\top) + \text{nullity}\,(A^\top) = m.$$

However, we have already seen that $\text{rank}\,(A) = \text{rank}\,(A^\top)$, so

$$\begin{aligned} m &= \text{rank}\,(A^\top) + \text{nullity}\,(A^\top) \\ &= \text{rank}\,(A) + \text{nullity}\,(A^\top). \end{aligned}$$

So if A is an $m \times n$ matrix with $\text{rank}\,(A) = r$, we have the following relationships:

$$\begin{array}{cc} \text{rank}\,(A) = \text{rank}\,(A^\top) = r & \\ \text{rank}\,(A) + \text{nullity}\,(A) = n & \text{rank}\,(A) + \text{nullity}\,(A^\top) = m \\ \text{nullity}\,(A) = n - r & \text{nullity}\,(A^\top) = m - r. \end{array}$$

Geometric Relationships Among the Fundamental Spaces

We have mentioned several times that, if A is an $m \times n$ matrix, then the vector spaces $\text{row}\,(A)$ and $\text{null}\,(A)$ are both subspaces of $\mathbb{R}^n$. Given this information, it makes sense to try to understand what relationships such as

$$\text{rank}\,(A) + \text{nullity}\,(A) = n$$

mean in terms of the *geometry* of Euclidean space.

Before we look at the details of the ideas, let's build some intuition by considering a simple example.

Example

Find $\text{row}\,(A)$ and $\text{null}\,(A)$, given

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

and describe the vector spaces geometrically.

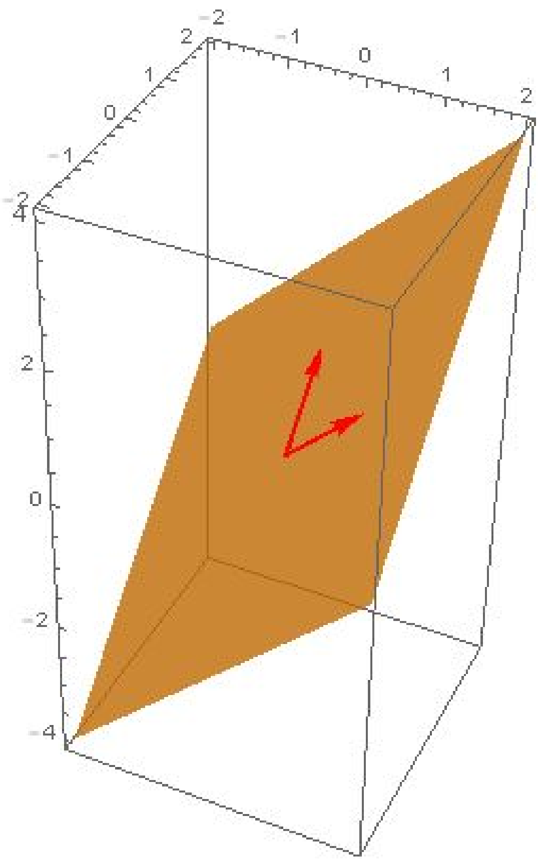
A is 2×3, so we know that the vector spaces $\text{row}\,(A)$ and $\text{null}\,(A)$ are both subspaces of $\mathbb{R}^3$, and we also know that

$$\text{rank}\,(A) + \text{nullity}\,(A) = 3.$$

Since A is already in row echelon form, it is easy to see that $\text{row}\,(A)$ is the two dimensional vector space spanned by vectors

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

These two vectors are graphed below, along with their span, which is a plane in $\mathbb{R}^3$:



Let's calculate the null space $\text{null}\,(A)$ of A: if $A\mathbf{x} = \mathbf{0}$, the augmented matrix for the equation is

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix};$$

we see that

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0, \end{aligned}$$

or

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3. \end{aligned}$$

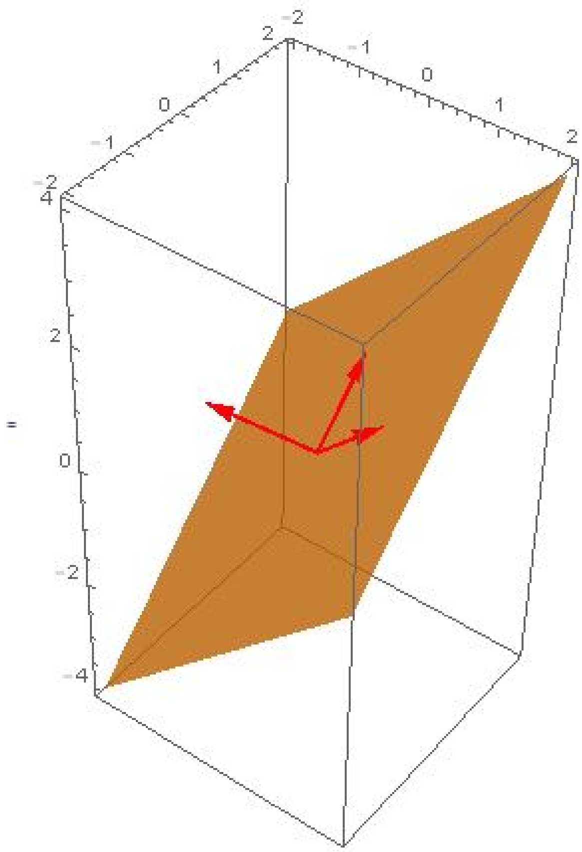
Thus we see that null (A) consists of all vectors of the form

$$\mathbf{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

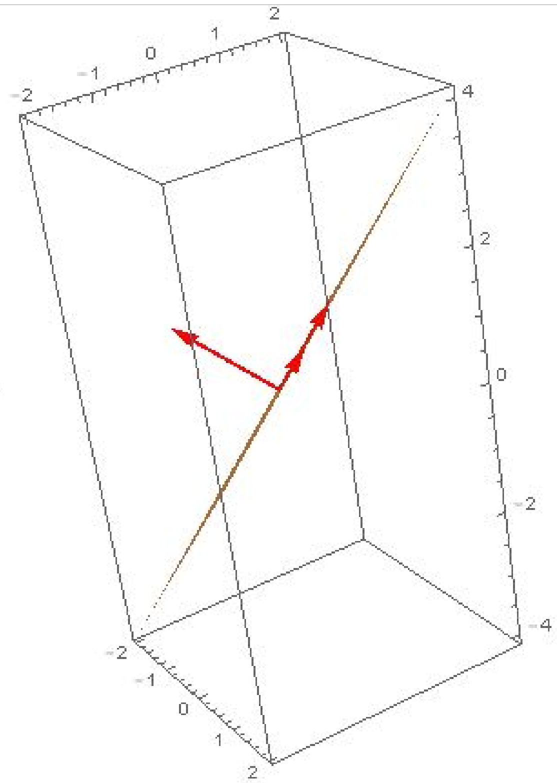
Thus null (A) is one-dimensional (as we expected, since rank $(A) = 2$ and nullity $(A) = 3 -$ rank $(A) = 1$), with basis

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

This basis vector is graphed below, along with the two basis vectors for row (A):



There is something interesting going on here; below is a rotated view of the same graph:



From the graph, it *appears* that the basis vector for null$\,(A)$ is orthogonal (perpendicular) to the plane, i.e. to the vectors in row$\,(A)$. We can check that this is true quite easily, using the idea of a *normal vector* from Calculus 3: the vector that results from calculating the "determinant" of the matrix

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

is said to be *normal* to the vectors

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

in the last two rows of the matrix; in particular, this vector is *orthogonal* to the plane formed by the span of $\mathbf{v_1}$ and $\mathbf{v_2}$. Let's make the calculation:

$$\begin{aligned}
\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} &= -\mathbf{i} - \mathbf{j} + \mathbf{k} \\
&= -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.
\end{aligned}$$

So the vector that forms the basis for null (A) (and indeed *every* vector in null (A)) is *orthogonal* to every vector in the vector space row (A)! We will see soon that this surprising result is actually true in general. Accordingly, we record a few relevant definitions below.

Orthogonal Complements

Definition 2. If W is a subspace of $\mathbb{R}^n$, the *orthogonal complement of* W, denoted $W^\perp$, is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W.

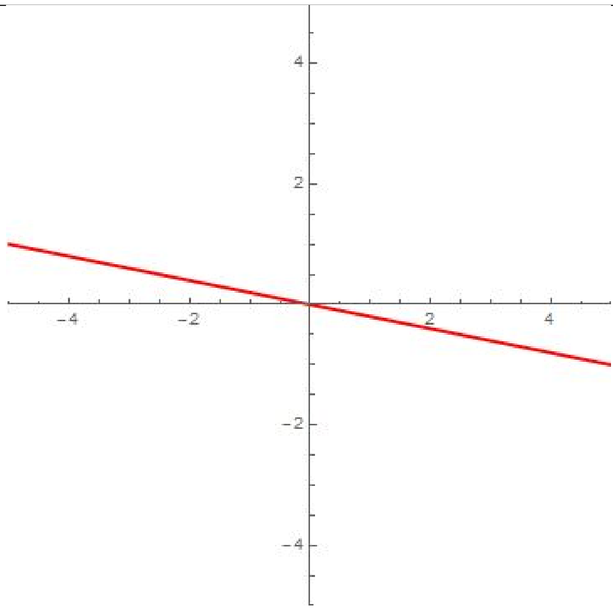
In terms of our example above, with

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{row}\,(A)$$

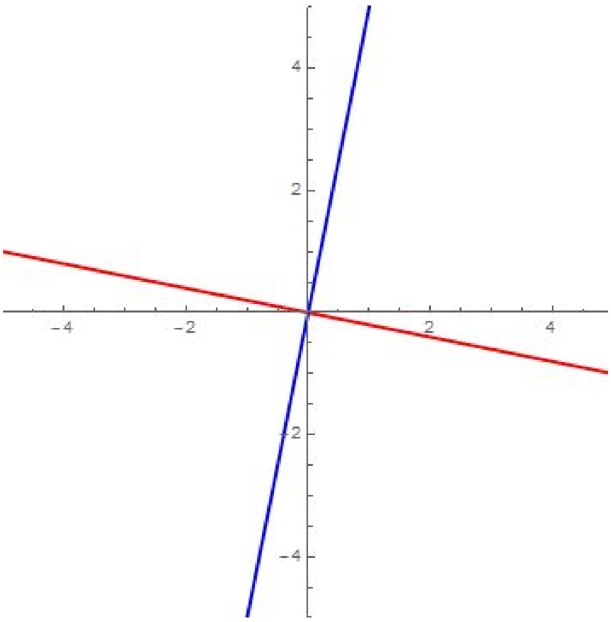
in $\mathbb{R}^3$, we see that the orthogonal complement of W in $\mathbb{R}^3$ is given by

$$W^\perp = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} = \text{null}\,(A).$$

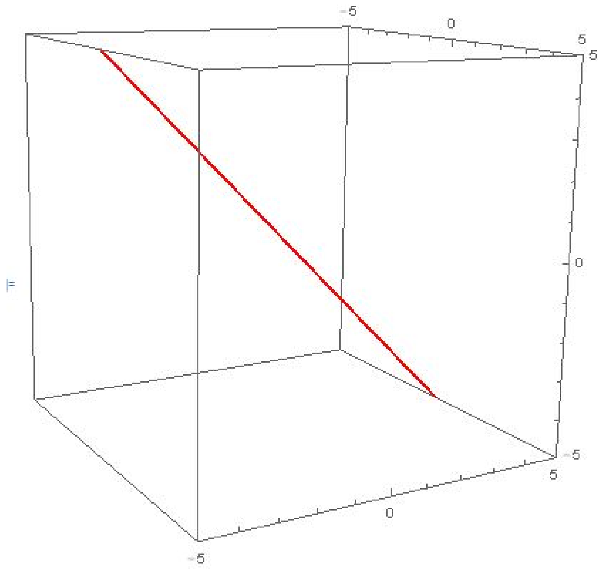
The red line graphed below is a subspace of $\mathbb{R}^2$:



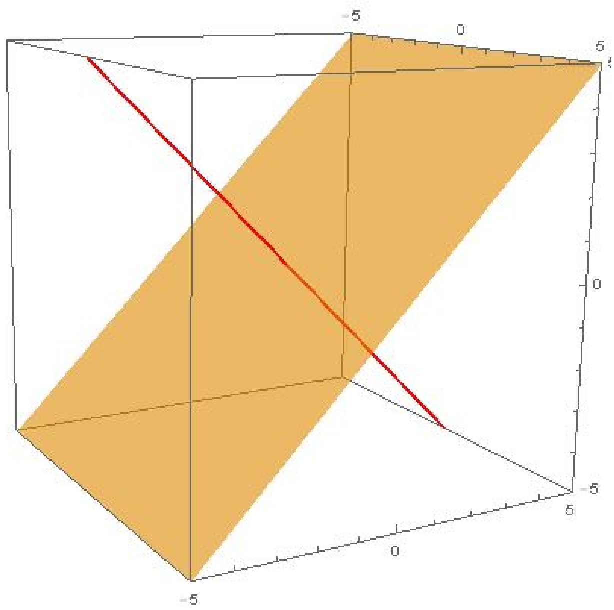
Its orthogonal complement is the blue line graphed below:



The line graphed in red below is a subspace of $\mathbb{R}^3$:



Its orthogonal complement is the plane graphed in orange below:



We record a few facts about orthogonal complements in the next theorem:

Theorem 4.8.6. If W is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. The only vector in both subspaces W and $W^\perp$ is $\mathbf{0}$

3. The orthogonal complement of the subspace $W^\perp$ is W.

Given our first example above, you may have already guessed the reason that orthogonal complements come up now in our discussion:

Theorem 4.8.7. If A is an $m \times n$ matrix, then:

(a) The null space null (A) and the row space row (A) of A are orthogonal complements in $\mathbb{R}^n$.

(b) The null space null $(A^\top)$ and the column space column (A) of A are orthogonal complements in $\mathbb{R}^m$.

This theorem is remarkable in the sense that it gives a relationship between the *algebraic structures* of two vector spaces and the *geometric structures* of the same vector spaces: if vector space W_1 spans the set of solutions to $A\mathbf{x} = \mathbf{b}$ and W_2 spans the set of solutions to $A\mathbf{x} = \mathbf{0}$, then *every vector in* W_1 *is orthogonal to every vector in* W_2.

We can now extend the list of equivalent conditions we have been maintaining:

Theorem 4.8.8. Let A be an $n \times n$ matrix. Then the following are equivalent:

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det A \neq 0$.

(h) The column vectors of A are linearly independent.

(i) The row vectors of A are linearly independent.

(j) The column vectors of A span $\mathbb{R}^n$ (column $(A) = \mathbb{R}^n$).

(k) The row vectors of A span $\mathbb{R}^n$ (row $(A) = \mathbb{R}^n$).

(l) The column vectors of A form a basis for $\mathbb{R}^n$.

(m) The row vectors of A form a basis for $\mathbb{R}^n$.

(n) rank $(A) = n$.

(o) nullity $(A) = 0$.

(p) The orthogonal complement of the null space of A is $\mathbb{R}^n$.

(q) The orthogonal complement of the row space of A is $\mathbf{0}$.