Section 4.7

Row Space, Column Space, and Null Space

Throughout this course, we have spent a great deal of time studying systems of equations such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

and the corresponding matrix equation

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

is the coefficient matrix,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is the vector of unknowns, and

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

is the vector of constants.

At this point, we have learned a great deal about the interconnections between the properties of the coefficient matrix A and the solution types. In fact, in Section 2.3, we saw the following list of equivalent conditions:

Theorem. Let A be an $n \times n$ matrix. Then the following are equivalent:

- A is invertible.
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A is a product of elementary matrices.
- $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

- $\det A \neq 0$.

The main point of the theorem is that (for square matrices) the coefficient matrix A determines whether *any* system $A\mathbf{x} = \mathbf{b}$ is consistent, and knowing data about A (in particular, its determinant) tells us immediately whether or not the system will be consistent.

If the coefficient matrix is not square, then the theorem does not tell us anything about the interconnections between data about A and solutions to the system $A\mathbf{x} = \mathbf{b}$. However, if you have guessed that there should still be *some* sort of interconnection, then you are absolutely correct; these interconnections are the topic of this section.

Row Space and Column Space

We will see in this and in the next section that the essential data about a matrix A has to do with the *row vectors* and *column vectors* that make up A:

Definition 1. Given an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

the vectors

$$\begin{aligned} \mathbf{r_1} &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix} \\ \mathbf{r_2} &= \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix} \\ &\vdots \\ \mathbf{r_m} &= \begin{pmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \end{aligned}$$

in $\mathbb{R}^n$ are called the *row vectors* of A, and the vectors

$$\mathbf{c_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \ \mathbf{c_2} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \ \dots, \ \mathbf{c_1} = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

in $\mathbb{R}^m$ are called the *column vectors* of A.

Example

The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is 3×2; thus it has three row vectors

$$\mathbf{r_1} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \mathbf{r_2} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \text{ and } \mathbf{r_3} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

all of which are in $\mathbb{R}^2$, and two column vectors

$$\mathbf{c_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{c_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

which are both vectors in $\mathbb{R}^3$.

The vectors which make up the rows and columns of A give us a great deal of information about the structure of A. Let's think about the geometric structure of these vectors, using the matrix and vectors from the example above.

The row vectors

$$\mathbf{r_1} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \mathbf{r_2} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \text{ and } \mathbf{r_3} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

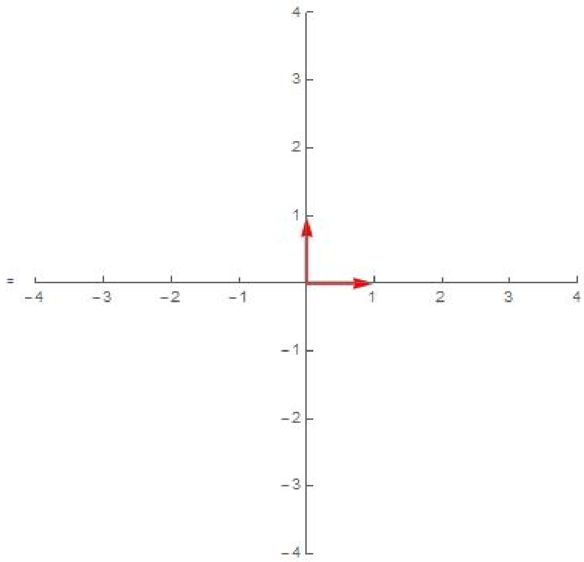
of matrix A are all in $\mathbb{R}^2$; let's think geometrically about them by considering their span

$$\text{span}\,\{\mathbf{r_1}, \ \mathbf{r_2}, \ \mathbf{r_3}\}.$$

Of course, $\mathbf{r_1}$ and $\mathbf{r_3}$ are the same vector, so this reduces to

$$\text{span}\,\{\mathbf{r_1}, \ \mathbf{r_2}\} = \text{span}\,\{\begin{pmatrix} 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \end{pmatrix}\}.$$

The vectors are graphed below in $\mathbb{R}^2$:



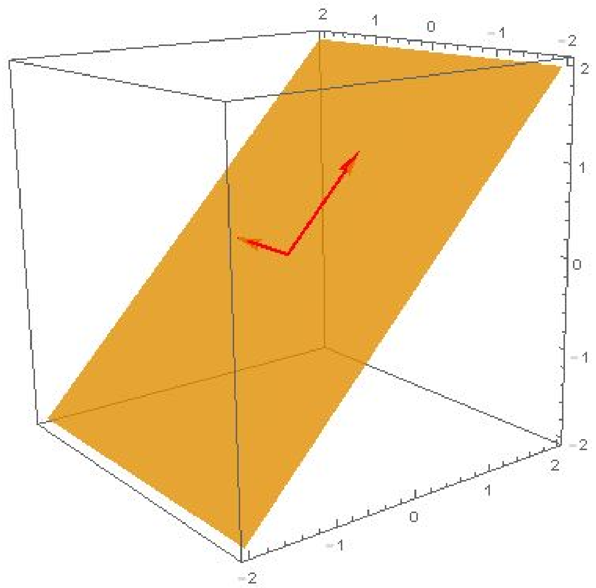
Of course, we know that the set $\{\mathbf{r_1}, \mathbf{r_2}\}$ is a basis for $\mathbb{R}^2$, so

$$\text{span}\,\{\mathbf{r_1},\ \mathbf{r_2}\} = \mathbb{R}^2.$$

On the other hand, the column vectors

$$\mathbf{c_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{c_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are in $\mathbb{R}^3$. They are graphed below:



Since $\mathbb{R}^3$ is three dimensional, the set $\{\mathbf{c_1},\ \mathbf{c_2}\}$ doesn't span $\mathbb{R}^3$; in fact, the span of the set is the (two dimensional) plane graphed above.

We will see in this and in the next section that the spans of the row vectors and column vectors of A give us a multitude of data about A. With this in mind, we define the *row space* and *column space* of A:

Definition 1. Given an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

its row vectors span a subspace of $\mathbb{R}^n$ called the *row space* of A, and we denote this vector space by $\text{row}\,(A)$. The column vectors of A span a subspace of $\mathbb{R}^m$ called the *column space* of A, which we denote by $\text{column}\,(A)$.

In the example above, we saw that the row space of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is all of $\mathbb{R}^2$, whereas the column space of A is a two dimensional subspace of $\mathbb{R}^3$.

In Section 4.2, we briefly mentioned the following theorem:

Theorem. If A is an $m \times n$ matrix, then the set of all solution vectors $\mathbf{x}$ to the equation

$$A\mathbf{x} = \mathbf{0}$$

is a subspace of $\mathbb{R}^n$.

We are now ready to study this solution set in more detail; indeed, we will see that this set is closely tied to the ideas of row space and column space. Accordingly, we now give this subspace a name:

Definition 1. The *null space* of an $m \times n$ matrix A, denoted null$\,(A)$, is the solution space of the system

$$A\mathbf{x} = \mathbf{0},$$

which is a subspace of $\mathbb{R}^n$.

Let's calculate the null space of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}:$$

we need to find the subspace of $\mathbb{R}^2$ that forms the solution space to

$$A\mathbf{x} = \mathbf{0}.$$

Since A is a 3×2 matrix, $\mathbf{x}$ is 2×1.

Let's start by calculating $A\mathbf{x}$:

$$\begin{aligned} A\mathbf{x} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + 0 \\ 0 + x_2 \\ x_1 + 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix}. \end{aligned}$$

So in order for the equation $A\mathbf{x} = \mathbf{0}$ to be satisfied, we see that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

clearly there is *only* one way to satisfy the equation–choose

$$x_1 = x_2 = 0.$$

Thus the vector

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the only one in the solution space for $A\mathbf{x} = \mathbf{0}$, so that the null space of A is just the trivial space $\{\mathbf{0}\}$ consisting of one vector.

For future reference, we should make a quick note here: we will see that there is a sense in which row space and null space are opposing forces; in our case, the row space of A "took up" all of $\mathbb{R}^2$, so that there was no room for the null space, which was forced to be trivial.

Assorted Theorems About Row Space, Column Space, and Null Space

In general (especially as the size of the matrices increases), calculating the row, column, and null space of a given matrix A could seem to be a daunting task. Fortunately, we collect some theorems in this section which can significantly reduce the difficulty inherent in the calculations.

Theorem 4.7.3. Elementary row operations do not change the null space of a matrix.

Theorem 4.7.4. Elementary row operations do not change the row space of a matrix.

Collectively, Theorems 4.7.3 and 4.7.4 say that, if two matrices A and B are row equivalent–we can get from A to B via a sequence of elementary row operations, and vice versa–then they have the same row space and the same null space. In terms of making calculations for row and null space, you should *always* apply Gauss-Jordan elimination to find the row echelon form of matrix A; it will always be easier to find the row and null spaces of a matrix in this simple form. Indeed, the following theorem describes this explicitly:

Theorem 4.7.5. If a matrix R is in row echelon form, then the row vectors with leading 1s form a basis for the row space of R (and for any matrix row equivalent to R), and the column vectors with leading 1s form a basis for the column space of R.

Remark. We should make one quick point here–while elementary row operations do not change the *row space* of a matrix, they *do* change its column space. If instead we applied elementary *column* operations to our matrix, its column space would stay the same but its row space would change.

Before we investigate some example, we need to consider one more theorem:

Theorem 4.7.6. If A and B are row equivalent matrices, then:

(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are as well.

(b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B for a basis for its column space.

Given the previous remark, Theorem 6 seems a bit tricky; you can interpret it as an affirmation of the fact that row operations do not change dimensions of spaces. In particular, row equivalent matrices A and B may have different column spaces, but their column spaces will have *the same dimension.*

Example

Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 4 & 0 \\ 2 & 6 & 2 & -10 & 0 \\ 3 & 9 & 2 & -18 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Find:

1. A basis for the vector space $\text{row}\,(A)$.
2. A basis for the null space $\text{null}\,(A)$.
3. A basis for the vector space $\text{column}\,(A)$.

Theorems $3 - 5$ tell us that the easiest way to solve this problem is by reducing A via Gauss-Jordan elimination. You should check that the matrix R below in row echelon form is row equivalent to A:

$$R = \begin{pmatrix} 1 & 3 & 1 & -5 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix R has the same row and null space as A; although it has a different column space than A, we can use theorem 6 to recover the column space.

1. *A basis for the vector space* $\text{row}\,(A)$*:* Theorem 5 tells us that the row vectors

$$\begin{aligned} \mathbf{r_1}' &= \begin{pmatrix} 1 & 3 & 1 & -5 & 0 \end{pmatrix} \\ \mathbf{r_2}' &= \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\ \mathbf{r_3}' &= \begin{pmatrix} 0 & 0 & 1 & 3 & -1 \end{pmatrix} \end{aligned}$$

with leading 1s form a basis for the row space of R, thus of A as well. Since there are three vectors in the basis for the row space, we see that the row space of A is a three dimensional subspace of $\mathbb{R}^5$.

2. *A basis for the null space* null(A)*:* Since A and R have the same null space, we will find the solution set to the system

$$R\mathbf{x} = \mathbf{0}.$$

Of course, R is already in row echelon form, so this is not hard to do. The augmented equation for the system is

$$\begin{pmatrix} 1 & 3 & 1 & -5 & 0 & | & 0 \\ 0 & 1 & 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & 3 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix};$$

the second and third rows tell us that we must have

$$\begin{aligned} x_2 &= -x_4 - x_5 \\ x_3 &= -3x_4 + x_5. \end{aligned}$$

In combination with the first row, we see that

$$\begin{aligned} x_1 &= -3x_2 - x_3 + 5x_4 \\ &= 3x_4 + 3x_5 + 3x_4 - x_5 + 5x_4 \\ &= 11x_4 + 2x_5. \end{aligned}$$

Thus the null space of A is made up of all of the vectors of the form

$$\mathbf{x} = \begin{pmatrix} 11x_4 + 2x_5 \\ -x_4 - x_5 \\ -3x_4 + x_5 \\ x_4 \\ x_5 \end{pmatrix}.$$

We can actually find a basis for this space quite easily by rewriting $\mathbf{x}$:

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} 11x_4 + 2x_5 \\ -x_4 - x_5 \\ -3x_4 + x_5 \\ x_4 \\ x_5 \end{pmatrix} \\ &= \begin{pmatrix} 11x_4 \\ -x_4 \\ -3x_4 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 2x_5 \\ -x_5 \\ x_5 \\ 0 \\ x_5 \end{pmatrix} \\ &= x_4 \begin{pmatrix} 11 \\ -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Now we are free to choose x_4 and x_5 in any way that we like; *any* vector of the form in the last line above is in the null space of A. In other words, the null space of A consists of all linear combinations of the vectors

$$\begin{pmatrix} 11 \\ -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix};$$

this is a basis for the null space of A, and we now see that the null space is a two dimensional subspace of $\mathbb{R}^5$.

3. *A basis for the vector space* column (A)*:* Theorem 5 tells us that the column vectors

$$\mathbf{c_1}' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{c_2}' = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \mathbf{c_3}' = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for the column space of R; clearly the column space is a three dimensional subspace of $\mathbb{R}^4$. While the vectors *do not* form a basis for the column space of A, we can use Theorem 4.7.6 to solve the problem: since the first three columns of R are linearly independent and span the column space of R, the corresponding columns of A are linearly independent and span the column space of A. Thus the vectors

$$\mathbf{c_1} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \ \mathbf{c_2} = \begin{pmatrix} 1 \\ 6 \\ 9 \\ 1 \end{pmatrix}, \text{ and } \mathbf{c_3} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

form a basis for the three dimensional subspace of $\mathbb{R}^4$ that is the column space of A.

Remark. Above, we saw that the row space of A was a three dimensional subspace of $\mathbb{R}^5$, and that the null space of A was a two dimensional subspace of $\mathbb{R}^5$. Again, we see row space and null space as "opposing" forces: the row space took up 3 dimensions of $\mathbb{R}^5$, leaving only two dimensions for the null space.

More on Solutions to $A\mathbf{x} = \mathbf{b}$

When we consider a system of equations whose matrix equation is given by

$$A\mathbf{x} = \mathbf{b},$$

we want to know if the system is consistent–i.e., is there at least one vector $\mathbf{x}$ that satisfies the equation?

Now that we know the definition of column space, we can utilize a lovely theorem to quickly answer the question above:

Theorem 4.7.1. A system of linear equations with matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the matrix $\mathbf{b}$ is in the column space of A.

Example

Given the system of equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

use Theorem 1 to determine if the system is consistent if

1. $$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2. $$\mathbf{b} = \begin{pmatrix} 6 \\ -8 \\ 6 \end{pmatrix}.$$

According to Theorem 1, we need to check to see if $\mathbf{b}$ is in the column space of A, which we calculated earlier as

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\};$$

clearly this is the set of all vectors of the form

$$s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where s and t are scalars.

1. The vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is definitely not in the column space of A–there are no scalars s and t so that

$$s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is inconsistent (no solution vectors $\mathbf{x}$).

2. This time we're in luck–the vector

$$\mathbf{b} = \begin{pmatrix} 6 \\ -8 \\ 6 \end{pmatrix}$$

is clearly in the column space of A, since it can be written as

$$\mathbf{b} = \begin{pmatrix} 6 \\ -8 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 8 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

So the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ -8 \\ 6 \end{pmatrix}$$

is definitely consistent; it has at least one solution vector, and may have infinitely many.