Section 4.5

The Dimension of a Vector Space

We have spent a great deal of time and effort on understanding the geometry of vector spaces, but we have not yet discussed an important geometric idea–that of the *size* of the space.

For example, think about the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Which one is "bigger"? We have not defined precisely what we mean by "bigger" or "smaller", but intuitively, you know that $\mathbb{R}^3$ is bigger.

Now, the set $\mathcal{M}_{22}$ of all 2×2 real matrices is also a vector space, so we could play the same game with it: which is the bigger vector space, $\mathcal{M}_{22}$ or $\mathbb{R}^2$? This time, intuition fails us, because we are not used to thinking of $\mathcal{M}_{22}$ spatially.

In this section, we will define precisely what we mean by the "size" of a vector space, thereby giving us the tools to answer such questions. In addition, we will see how the size of a vector space is closely related to linear independence and spanning.

Base Size and Dimension

In Section 4.4, we saw that the set

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^2$. We also saw that the set

$$B_2 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

forms a basis for $\mathbb{R}^2$ as well.

It is interesting to note that both of the bases that we have found for $\mathbb{R}^2$ have *two* elements. Of course, there are *many* other bases for $\mathbb{R}^2$; is it possible to find a basis containing, say 3 vectors? Or a basis with only 1 vector? The following theorem answers this question in a surprising and mathematically beautiful way:

Theorem 4.5.1. If a vector space V has a basis consisting of n elements, then *every* basis for V has n elements.

Theorem 4.5.1 answers the question above in the negative: since we know of a basis for $\mathbb{R}^2$ that has two elements, then *any* basis that we can find for $\mathbb{R}^2$ has to have 2 elements. Similarly, we know that the set $\mathcal{M}_{22}$ of all real 2×2 matrices has a basis consisting of the matrices

$$\mathbf{e_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{e_{12}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \mathbf{e_{21}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \mathbf{e_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

the theorem says that *any* other basis you can find for $\mathcal{M}_{22}$ will also have 4 elements.

The theorem now gives us a precise way to define what we mean when we refer to the size of a vector space:

Definition 1. The *dimension* of a vector space V, denoted $\dim(V)$, is the number of vectors in a basis for V. We define the dimension of the vector space containing only the zero vector $\mathbf{0}$ to be 0.

In a sense, the dimension of a vector space tells us how many vectors are needed to "build" the space, thus gives us a way to compare the relative sizes of the spaces. We have already seen that

- $\dim(\mathbb{R}^2) = 2$
- $\dim(\mathbb{R}^3) = 3$
- $\dim(\mathcal{M}_{22}) = 4$,

so our original observation that $\mathbb{R}^3$ is a "larger" space than is $\mathbb{R}^2$ is correct (and now defined more precisely). Similarly, we can conclude that $\mathcal{M}_{22}$ is a larger space that $\mathbb{R}^3$, as it takes more vectors to build the space.

Below is a list of the dimensions of some of the vector spaces that we have discussed frequently. Recall that $\mathcal{M}_{mn}$ refers to the vector space of $m \times n$ matrices; P_n refers to the vector space of polynomials of degree no more than n; and U_2 refers to the vector space of 2×2 upper triangular matrices.

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathcal{M}_{mn}) = m \cdot n$
- $\dim(P_n) = n + 1$
- $\dim(U_2) = 3$

Understanding Bases

In section 4.4, we noted that a basis for a vector space V has to have *enough* vectors to be able to span V, but not so many that it is no longer linearly independent. We make this idea precise with the following theorem:

Theorem 4.5.2. Let V be an n-dimensional vector space, that is, every basis of V consists of n vectors. Then

(a) Any set of vectors from V containing *more than* n vectors is linearly dependent.

(b) Any set of vectors from V containing *fewer than* n vectors does not span V.

Key Point. Adding too many vectors to a set will force the set to be linearly dependent; on the other hand, taking too many vectors away from a set will prevent it from spanning. A basis set is a sort of "sweet spot" between linear independence and spanning: if a basis S for V has n elements and we remove one, then S no longer spans; add one, and S is no longer linearly independent.

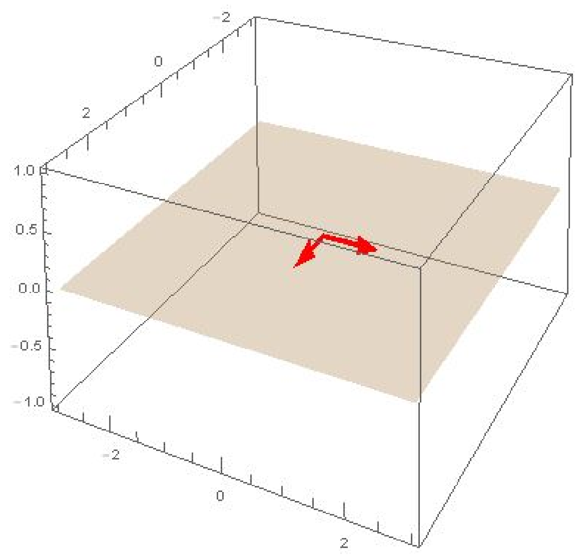
As an example, we know that the set

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$; it is linearly independent, and it spans $\mathbb{R}^3$. If we remove the last vector, to get the set

$$B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

then we no longer have a spanning set for $\mathbb{R}^3$. Indeed, this is easy to see geometrically: the two vectors from B' are graphed below in $\mathbb{R}^3$, and clearly cannot be used to describe the vertical component of the space:



In fact, these two vectors span the *subspace* of $\mathbb{R}^3$ shaded in gray, but not all of $\mathbb{R}^3$.

The theorems above lead to some important facts about the geometry of vector spaces and their subspaces.

Theorem 4.5.3. *(Pus/Minus Theorem)* Let S be a nonempty set of vectors in a vector space V.

(a) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that lies outside span (S), the the set

$$S \cup \{\mathbf{v}\}$$

of all of the vectors in S in addition to $\mathbf{v}$ is still linearly independent.

(b) If $\mathbf{v}$ is a vector in S that is a linear combination of some of the other vectors in S, then the set

$$S - \{\mathbf{v}\}$$

of all of the vectors in S except for $\mathbf{v}$ spans the same subspace of V as that spanned by S, that is

$$\text{span}\,(S - \{\mathbf{v}\}) = \text{span}\,(S).$$

In essence, part (b) of the theorem says that, if a set is linearly dependent, then we can remove excess vectors from the set *without* affecting the set's span.

We will discuss part (a) Theorem 3 in more detail momentarily; first, let's look at an immediate consequence of the theorem:

Theorem 4.4.5. Let V be an n-dimensional vector space, and let S be a set of n vectors in V. If either

- S is linearly independent, *or*
- S spans V,

then S is a basis for V.

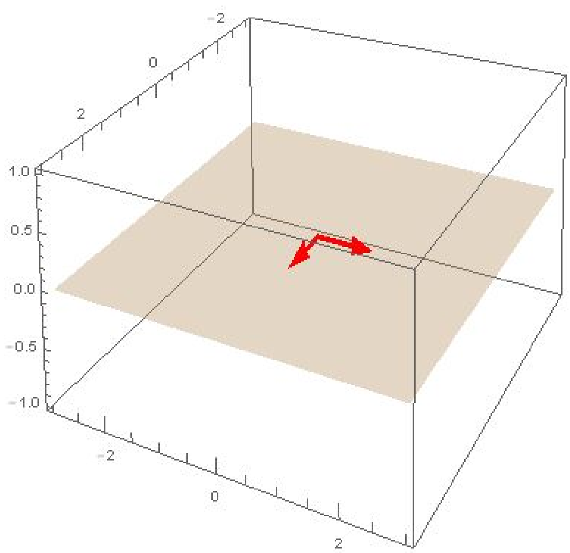
Key Point. We know that a basis for a vector space must be a linearly independent spanning set; to conclude that a set is a basis, we must be certain that both conditions are met.

However, Theorem 4.4.5 makes it much easier to determine whether or not a set is a basis: if a set has the right number of vectors–the same as the dimension of V–then we can quickly check to see if the set is a basis by determining if it is a linearly independent set, or alternatively by checking that the set spans V. We don't have to check *both* conditions anymore, just one of them!

Let's use Theorems 3 and 4 to investigate the example introduced above: the set

$$B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

while linearly independent in $\mathbb{R}^3$, *does not* span $\mathbb{R}^3$:



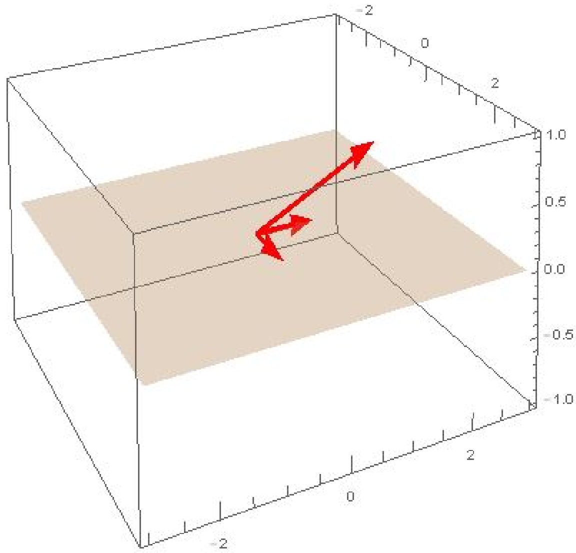
Now part (a) of Theorem 3 says that *If S is a linearly independent set, and if* $\mathbf{v}$ *is a vector in V that lies outside* $\operatorname{span}(S)$*, then the set*

$$S \cup \{\mathbf{v}\}$$

of all of the vectors in S in addition to $\mathbf{v}$ *is still linearly independent.*

In other words, we can add *any* vector we like to B' (as long as that vector is not already in the span of B'), and we will still have a linearly independent set. In this case, we just need to choose a vector that does not lie in the xy plane–for example, we could choose

$$\bigl(2, 1, 1\bigr) :$$



The Theorem 4 assures us that the set

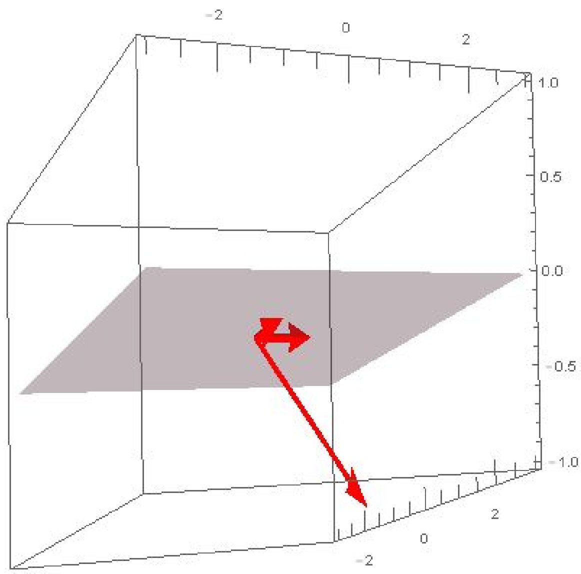
$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is still linearly independent–and in this particular example, a basis for $\mathbb{R}^3$, as $\mathbb{R}^3$ is 3 dimensional. Since the set is independent and has the right number of vectors, Theorem 4 tells us that we don't have to check that it spans $\mathbb{R}^3$ to know that it's a basis!

Alternatively, we could choose the vector

$$\bigl(0, 3, -1\bigr)$$

to add to B':



Again, the theorems tells us that the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right\}$$

is linearly independent (and a basis for $\mathbb{R}^3$, since it once again has 3 vectors).

Example

The vectors

$$\mathbf{f_1}(x) = 2x - 3, \ \mathbf{f_2}(x) = x^2 + 1, \text{ and } \mathbf{f_3}(x) = 2x^2 - x$$

are linearly independent. Complete the set to form a basis for P_3, the set of all polynomials of degree no more than 3.

We know that a basis for P_3 must consist of 4 linearly independent vectors that span P_3. Since we already have 3 linearly independent vectors, we simply need to find one more vector to add to the set.

Now Theorem 3 tells us that, if $\mathbf{f_4}$ is linearly independent from $\mathbf{f_1}(x)$, $\mathbf{f_2}(x)$, and $\mathbf{f_3}(x)$, then the set

$$\{\mathbf{f_1}, \ \mathbf{f_2}, \ \mathbf{f_3}, \ \mathbf{f_4}\}$$

will also be linearly independent. In other words, we will have a set of 4 linearly independent vectors in a 4-dimensional space–Theorem 4 tells us that this will be a basis.

So our only remaining task is to find a vector linearly independent from $\mathbf{f_1}(x)$, $\mathbf{f_2}(x)$, and $\mathbf{f_3}(x)$. This is not too difficult to do–since none of the polynomials in our list include a term of x^3, $\mathbf{f_4}(x) = x^3$ seems like a good choice. We can check to make sure that the functions are indeed linearly independent by calculating their Wronskian:

$$\begin{aligned}
W &= \det\begin{pmatrix} \mathbf{f_1}(x) & \mathbf{f_2}(x) & \mathbf{f_3}(x) & \mathbf{f_4}(x) \\ \mathbf{f_1}'(x) & \mathbf{f_2}'(x) & \mathbf{f_3}'(x) & \mathbf{f_4}'(x) \\ \mathbf{f_1}''(x) & \mathbf{f_2}''(x) & \mathbf{f_3}''(x) & \mathbf{f_4}''(x) \\ \mathbf{f_1}'''(x) & \mathbf{f_2}'''(x) & \mathbf{f_3}'''(x) & \mathbf{f_4}'''(x) \end{pmatrix} \\
&= \det\begin{pmatrix} 2x-3 & x^2+1 & 2x^2-x & x^3 \\ 2 & 2x & 4x-1 & 3x^2 \\ 0 & 2 & 4 & 6x \\ 0 & 0 & 0 & 6 \end{pmatrix} \\
\text{(cofactor expansion, row 4)} &= 6\det\begin{pmatrix} 2x-3 & x^2+1 & 2x^2-x \\ 2 & 2x & 4x-1 \\ 0 & 2 & 4 \end{pmatrix} \\
\text{(cofactor expansion, column 1)} &= 6\left((2x-3)\det\begin{pmatrix} 2x & 4x-1 \\ 2 & 4 \end{pmatrix} - 2\det\begin{pmatrix} x^2+1 & 2x^2-x \\ 2 & 4 \end{pmatrix}\right) \\
&= 6((2x-3)(8x-8x+2) - 2(4x^2+4-4x^2+2x)) \\
&= 6(4x-6-8-4x) \\
&= -84.
\end{aligned}$$

Since the Wronskian of these functions is $-84 \neq 0$, they are linearly independent.

By Theorem 4, a linearly independent set of 4 vectors in a four dimensional vector space is a basis for the space; thus our set

$$\{2x-3,\ x^2+1,\ 2x^2-x,\ x^3\}$$

is a basis.

More Consequences of the Basis Theorems

We now present a few more theorems about the interconnections among spanning sets, linearly independent sets, and bases. Again, we see the idea that spanning sets are relatively large sets, and independent sets are relatively small:

Theorem 4.5.5. Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis of V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis by inserting appropriate vectors into S.

Theorem 4.5.6. If W is a subspace of a (finite-dimensional) vector space V, then:

(b) $\dim W \leq \dim V$

(c) $\dim W = \dim V$ if and only if $W = V$.

Theorem 6 tells us that, if we reduce the number of vectors in a vector space, we automatically reduce the dimension of the space.