Section 3.1

Properties of Euclidean Space

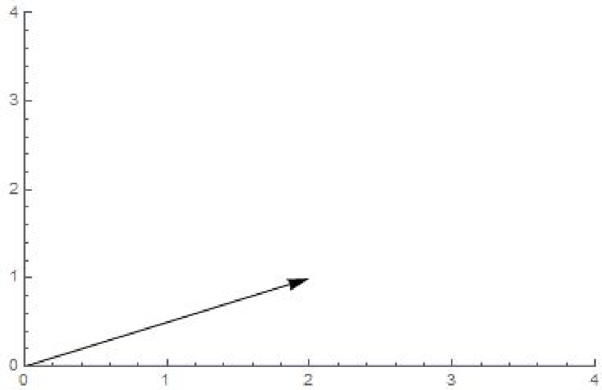
As has been our convention throughout this course, we use the notation $\mathbb{R}^2$ to refer to the plane (two dimensional space); $\mathbb{R}^3$ for three dimensional space; and $\mathbb{R}^n$ to indicate n dimensional space. Alternatively, these spaces are often referred to as *Euclidean spaces*; for example, "three dimensional Euclidean space" refers to $\mathbb{R}^3$.

In this section, we will point out many of the special features common to the Euclidean spaces; in Chapter 4, we will see that many of these features are shared by other "spaces" as well.

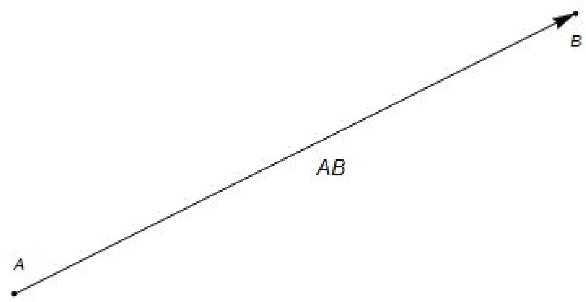
Many of the graphics in this section are drawn in two dimensions (largely because this is the easiest space to visualize on paper), but you should be aware that the ideas presented apply to n dimensional Euclidean space, not just to two dimensional space.

Vectors in Euclidean Space

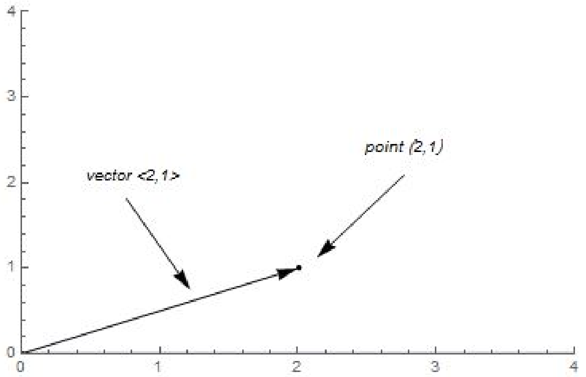
When we refer to a *vector in Euclidean space*, we mean directed line segments that are embedded in the space, such as the vector pictured below in $\mathbb{R}^2$:



We often refer to a vector, such as the vector **AB** shown below, by its **initial** and **terminal points**; in this example, A is the initial point of **AB**, and B is the terminal point of the vector.

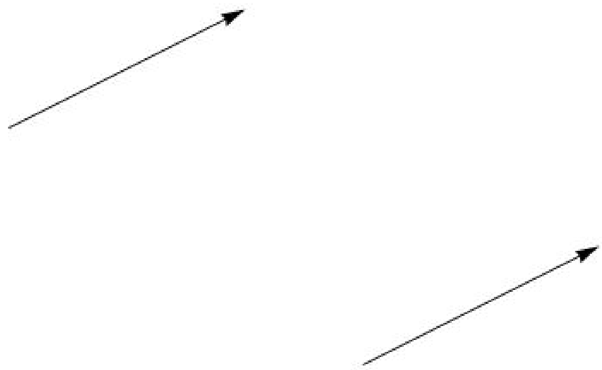


While we often think of n dimensional space as being made up of *points*, we may equivalently consider it to be made up of vectors by identifying points and vectors. For instance, we identify the point $(2, 1)$ in two dimensional Euclidean space with the vector with initial point $(0, 0)$ and terminal point $(2, 1)$:



In this way, we think of n dimensional Euclidean space as being made up of n dimensional vectors.

A vector is completely determined by its length and direction, so that two vectors with the same length and direction are said to be *equal* or *equivalent*, even when they are located in different areas of space. The vectors in the following diagram are equal:



Equivalent vectors.

The vector whose initial and terminal points are the same is called the *zero vector*, and is denoted by $\mathbf{0}$.

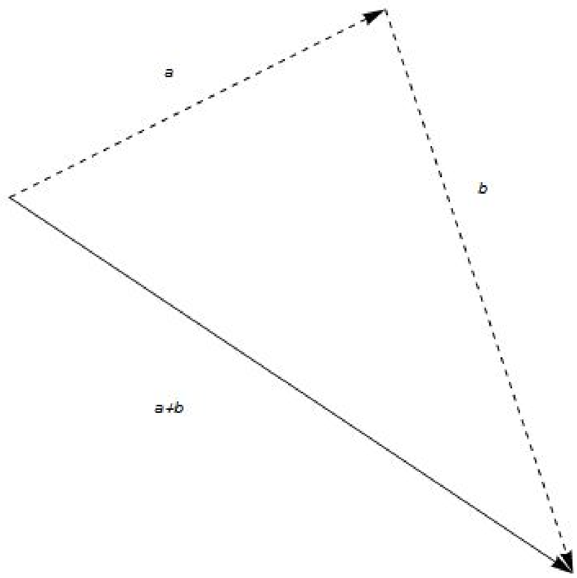
Key Point. It is important to note here that we think of the vectors that make up Euclidean space as *directed line segments*, but in the context of Chapter 4 we will see other types of vectors that don't correspond to "arrows" in space. You should try to divorce yourself from the idea that "all vectors are directed line segments," as this interpretation really only makes sense in Euclidean space.

Operations on Vectors

There are several ways to use given vectors to build more vectors. The first *operation* that we will discuss is *vector addition*:

Definition. Let **a** and **b** be vectors. The vector whose initial point is the same as the initial point of **a**, and whose terminal point is the terminal point of **b** when the initial point of **b** is placed at the terminal point of **a** is called the *sum* of **a** and **b**, and is denoted by $\mathbf{a} + \mathbf{b}$.

The vector $\mathbf{a} + \mathbf{b}$ which is the sum of the two given vectors **a** and **b** is indicated below:



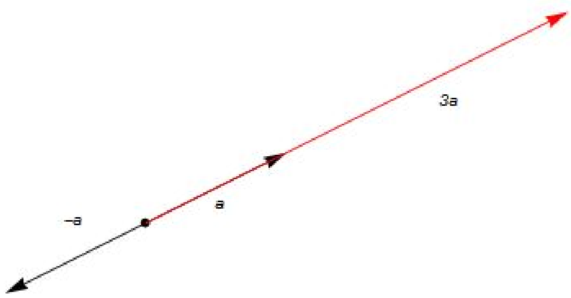
Another important operation is *scalar multiplication*:

Definition. Let **a** be a vector and k be a nonzero scalar (for our purposes, a real number). The vector $c\mathbf{a}$ whose length is $|c|$ times the length of **a** and whose direction is

- the same as the direction of **a** if $k > 0$, or
- the opposite direction from **a** if $k < 0$

is called the *scalar product of* **a** *by* k.

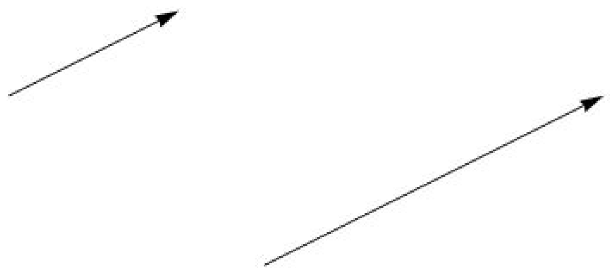
The vectors **a**, $-\mathbf{a}$, and $3\mathbf{a}$ are graphed below:



The graph above leads to an important idea; note that all three of the vectors lie on the same line. In general, we say that

Definition. Two vectors are *collinear* if they can be translated so that they lie on the same line.

For example, the vectors in the graph below are collinear. They do not currently lie on the same line, but we could certainly shift one of them over so that they do.



Component Form

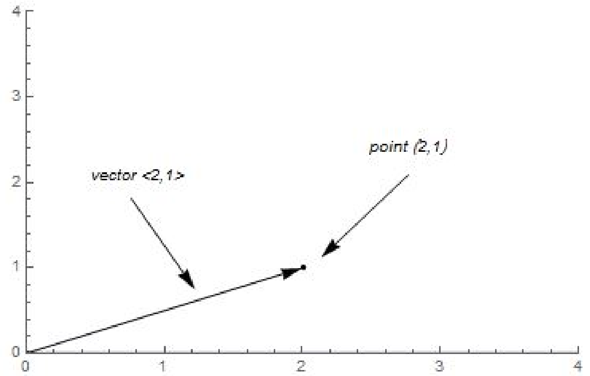
It is often convenient to refer to a vector in its *component form.*

Definition. The *component form*

$$\langle v_1, v_2, \ldots, v_n \rangle$$

of a vector $\mathbf{v}$ in n dimensional Euclidean space indicates that, when the initial point of $\mathbf{v}$ is placed at the origin $(0, 0, \ldots, 0)$, the terminal point of $\mathbf{v}$ falls at the point $(v_1, v_2, \ldots, v_n)$.

For example, the vector indicated below is positioned so that its initial point is at $(0, 0)$; its terminal point is at $(2, 1)$.



We may refer to this vector using the notation $\langle 2, 1 \rangle$.
Similarly, the notation

$$\langle 2, 0, 2, -1 \rangle$$

refers to the vector in *four dimensional space* whose terminal point falls at $(2, 0, 2, -1)$ when its initial point is placed at the origin $(0, 0, 0, 0)$.

The component form of the zero vector in n dimensional space is given by

$$\mathbf{0} = \langle 0, 0, \ldots, 0 \rangle.$$

The ideas of vector equality, vector addition, and scalar multiplication can easily be rewritten in terms of components:

Definition 2. Two vectors

$$\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle \text{ and } \mathbf{w} = \langle w_1, w_2, \ldots, w_n \rangle$$

in n dimensional space are *equal* if their component forms are the same, that is if

$$v_1 = w_1,\ v_2 = w_2, \ldots, \text{ and } v_n = w_n.$$

Definition 3. The *sum* $\mathbf{v} + \mathbf{w}$ of two vectors

$$\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle \text{ and } \mathbf{w} = \langle w_1, w_2, \ldots, w_n \rangle$$

in n dimensional space is the vector whose component form is given by

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle.$$

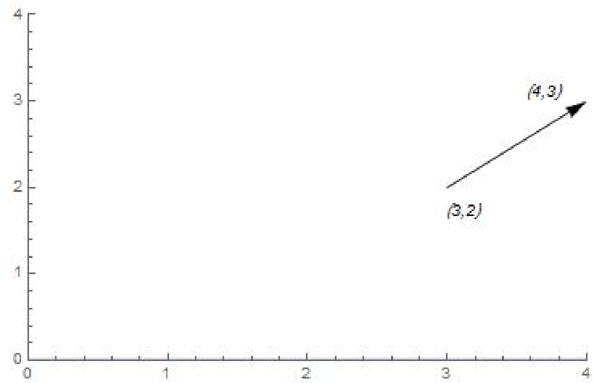
Definition 3. The *scalar product* $k\mathbf{v}$ of a vector

$$\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$$

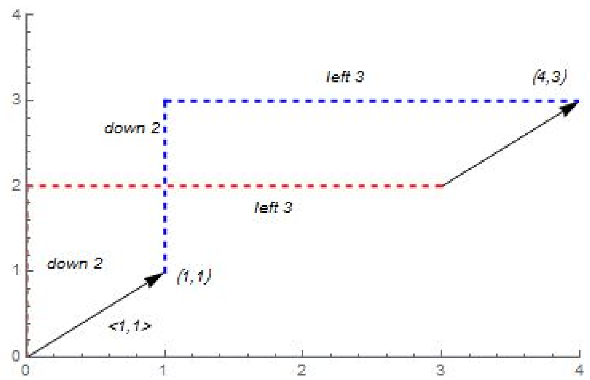
in n dimensional space with the constant k is the vector whose component form is given by

$$k\mathbf{v} = \langle kv_1, kv_2, \ldots, kv_n \rangle.$$

Even if the initial point of a particular vector is *not* at the origin, it is quite easy to calculate its component form. As an example, let's find the component form of the vector graphed below, whose initial point is at $(3, 2)$, and whose terminal point is at $(4, 3)$:



To make the calculation, we simply need to know where the terminal point of this vector would lie if we were to move its initial point back to the origin. In other words, we need to move the entire vector *left* by 3 units, and *down* by 2 units:



Thus the component form of the vector is

$$\langle 1, 1 \rangle.$$

The same idea holds for finding component forms of vectors in higher dimensional Euclidean space:

Key Point. The component form of a vector $\mathbf{v}$ whose initial point and terminal point lie at

$$(w_1, w_2, \ldots, w_n) \text{ and } (v_1, v_2, \ldots, v_n),$$

respectively, is given by

$$\mathbf{v} = \langle v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n \rangle.$$

Example

Let $\mathbf{v}$ be the vector with initial point and terminal point at $(2, 10, -4, 3)$ and $(-1, 5, 2, 2)$, respectively, and let $\mathbf{w}$ and $\mathbf{u}$ be the vectors with component forms

$$\mathbf{w} = \langle 1, 0, 3, 1 \rangle \ \mathbf{u} = \langle -3, -5, 6, -1 \rangle.$$

1. Find the component form of $\mathbf{v}$.
2. Determine if any of the vectors are equal.
3. Find $\mathbf{v} + \mathbf{w}$.
4. Find $-2\mathbf{u}$.

1. To put $\mathbf{v}$ in component form, we need to subtract the coordinates of the initial point of $\mathbf{v}$ from the coordinates of its terminal point; thus we see that

$$\mathbf{v} = \langle -1 - 2, 5 - 10, 2 + 4, 2 - 3 \rangle = \langle -3, -5, 6, -1 \rangle.$$

2. Now that each of the vectors is in component form, it is easy to compare their coordinates to determine equality. Clearly $\mathbf{v}$ and $\mathbf{w}$ have different component forms, thus are not the same vector; however, $\mathbf{v}$ and $\mathbf{u}$ have the same component form, so that

$$\mathbf{v} = \mathbf{u}.$$

3. We calculate $\mathbf{v} + \mathbf{w}$ by adding corresponding coordinates form the component forms of the vectors, so that

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle -3+1, -5+0, 6+3, -1+1 \rangle \\ &= \langle -2, -5, 9, 0 \rangle. \end{aligned}$$

4. To calculate $-2\mathbf{u}$, we simply need to multiply each of the coordinates of $\mathbf{u}$ by the scalar -2. So

$$-2\mathbf{u} = \langle -2 \cdot (-3), -2 \cdot (-5), -2 \cdot 6, -2 \cdot (-1) \rangle = \langle 6, 10, -12, 2 \rangle.$$

Properties of Vector Operations in Euclidean Space

As mentioned at the beginning of this section, the various Euclidean spaces share properties that will be of significance in our study of linear algebra. Many of these properties are listed in the following theorem:

Theorem 3.1.1. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in n dimensional Euclidean space, and k and m are scalars (real numbers), then:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(f) $(k+m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(g) $(km)\mathbf{u} = k(m\mathbf{u})$

(h) $1\mathbf{u} = \mathbf{u}$

These properties of vector addition and scalar multiplication say that the Euclidean spaces of various dimensions are quite "nice", and behave similarly. For example, even though $\mathbb{R}^2$ and $\mathbb{R}^5$ are different spaces, the behavior of the vectors that make them up are similar: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$, regardless of whether the two vectors are in $\mathbb{R}^2$, $\mathbb{R}^5$, or, say $\mathbb{R}^{100}$.

Linear Combinations

Just as we can create linear combinations of matrices, we can create linear combinations of vectors:

Definition 4. The vector $\mathbf{w}$ in $\mathbb{R}^n$ is called a *linear combination* of vectors $\mathbf{v_1}$, $\mathbf{v_2}$,...,$\mathbf{v_r}$ in $\mathbb{R}^n$ if $\mathbf{w}$ can be written as a sum of scalar multiples of these vectors, i.e. if there are real numbers k_1, k_2,...,k_r so that

$$\mathbf{w} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots + k_r\mathbf{v_r}.$$

In a sense, saying that vector $\mathbf{w}$ is a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$,...,$\mathbf{v_r}$ is another way of saying that we can use the vectors $\mathbf{v_1}$, $\mathbf{v_2}$,...,$\mathbf{v_r}$, along with the operations of scalar multiplication and vector addition, to "build" $\mathbf{w}$.

Example
Show that

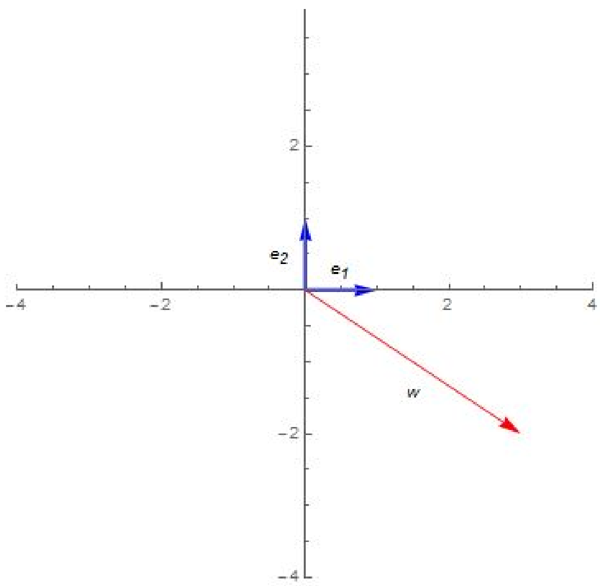
$$\mathbf{w} = \langle 3, -2 \rangle$$

is a linear combination of the standard basis vectors

$$\mathbf{e_1} = \langle 1, 0 \rangle, \text{ and } \mathbf{e_2} = \langle 0, 1 \rangle$$

in $\mathbb{R}^2$.

The three vectors in question are graphed below:



To show that $\mathbf{w}$ is a linear combination of $\mathbf{e_1}$ and $\mathbf{e_2}$, we need to find scalars k_1 and k_2 so that we can write

$$\mathbf{w} = k_1\mathbf{e_1} + k_2\mathbf{e_2}.$$

Of course, since we know the component forms of the three vectors, this is quite simple to do: we need 3 copies of $\mathbf{e_1}$ and -2 copies of $\mathbf{e_2}$:

$$\begin{aligned} 3\mathbf{e_1} - 2\mathbf{e_2} &= 3\langle 1, 0\rangle - 2\langle 0, 1\rangle \\ &= \langle 3, 0\rangle + \langle 0, -2\rangle \\ &= \langle 3, -2\rangle \\ &= \mathbf{w}. \end{aligned}$$

So the desired linear combination is given by

$$\mathbf{w} = 3\mathbf{e_1} - 2\mathbf{e_2}.$$

Graphically, we see that we can indeed use $\mathbf{e_1}$ and $\mathbf{e_2}$ to "build" $\mathbf{w_1}$: