Diagonal, Triangular, and Symmetric Matrices

Certain matrices, such as the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

have a special "shape", which endows the matrix with special properties. The identity matrix is an example of a *diagonal* matrix; we will discuss diagonal, triangular, and symmetric matrices and their properties in this section.

Diagonal Matrices

Examine the matrices below:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{pmatrix}, \quad \begin{pmatrix} .5 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

You should notice that the three matrices have a common shape–their *only* nonzero entries occur on their diagonals.

Definition. A square matrix is *diagonal* if all of its off-diagonal entries are 0s. A diagonal matrix has form

$$\begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}.$$

We give such matrices a name because they have interesting properties not shared by non-diagonal matrices; we discuss these properties below.

Properties of Diagonal Matrices

We have seen that matrix multiplication is, in general, quite tedious. However, if the two matrices in question are actually *diagonal* matrices, multiplication becomes quite simple. To understand why, let's try a simple example:

Example. Calculate the product

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Recall that the a_{ij} entry of a product of two matrices is the scalar product of the ith row of the first matrix and the jth column of the second; so the $1,1$ entry of the product should be

$$a_{11} = -2 \cdot 4 + 0 + 0 = -8;$$

thus the product has form

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & & \\ & & \\ & & \end{pmatrix}.$$

Let's calculate the next entry:

$$a_{12} = -2 \cdot 0 + 0 \cdot (-1) + 0 = 0$$

(easy to calculate since there's at least one zero in each term!). Thus the product takes on the form

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 0 & \\ & & \\ & & \end{pmatrix}.$$

We should be able to calculate the remaining entries a bit more cleverly–as you may have noticed, the large numbers of 0s in the original matrices makes the calculations quite simple. Indeed, since we are multiplying rows by columns, we simply need to locate the rows and columns which have *nonzero* entries that get multiplied together. Let's think about the $1,3$ entry, which we get by multiplying the highlighted row and column:

$$\begin{pmatrix} \color{red}{-2} & \color{red}{0} & \color{red}{0} \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & \color{red}{0} \\ 0 & -1 & \color{red}{0} \\ 0 & 0 & \color{red}{-2} \end{pmatrix}.$$

Clearly $a_{13} = 0$, as each term of the scalar product includes at least one 0.

The same phenomena occurs for the $2,1$ entry–no nonzero entries "match up":

$$\begin{pmatrix} -2 & 0 & 0 \\ \color{red}{0} & \color{red}{5} & \color{red}{0} \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} \color{red}{4} & 0 & 0 \\ \color{red}{0} & -1 & 0 \\ \color{red}{0} & 0 & -2 \end{pmatrix}.$$

Let's try the $2,2$ entry:

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

This example is different–the 5 and -1 "match up", so that $a_{22} = -5$.
Let's look at our results so far. We've seen that

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -5 & \\ & & \end{pmatrix}.$$

At this point, you may have guessed the punch line–off-diagonal entries will be 0, and diagonal entries are the products of the corresponding entries from the original matrices. Indeed, the final matrix product is

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

The pattern we saw in the previous example holds true in general, as indicated by the following theorem:

Theorem. The product of a pair of diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ 0 & 0 & b_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_n \end{pmatrix}$$

is also a diagonal matrix, and has form

$$AB = \begin{pmatrix} a_1b_1 & 0 & 0 & \dots & 0 \\ 0 & a_2b_2 & 0 & \dots & 0 \\ 0 & 0 & a_3b_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_nb_n \end{pmatrix}.$$

The theorem above has several nice consequences. For starters, finding the inverse of a diagonal matrix is quite simple (unlike finding inverses for most other matrices!). Indeed, a diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; if this is the case, then the inverse of

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix} \text{ is } D^{-1} = \begin{pmatrix} d_1^{-1} & 0 & 0 & \dots & 0 \\ 0 & d_2^{-1} & 0 & \dots & 0 \\ 0 & 0 & d_3^{-1} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n^{-1} \end{pmatrix}.$$

Example. Given

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{4} \end{pmatrix},$$

find A^{-1} and A^3.

1. A^{-1} is simple, and you should check that

$$A^{-1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{5} \end{pmatrix}$$

2. To find A^3, we could start by finding A^2, then calculating $A^2 \cdot A$. However, we know that each product will yield another diagonal matrix, whose diagonal entries are just the products of the corresponding diagonal entries of the factors. So to find A^3, all we really need to do is cube each of its diagonal entries. We have

$$A^3 = \begin{pmatrix} (\frac{1}{2})^3 & 0 & 0 & 0 & 0 \\ 0 & 7^3 & 0 & 0 & 0 \\ 0 & 0 & 3^3 & 0 & 0 \\ 0 & 0 & 0 & (-1)^3 & 0 \\ 0 & 0 & 0 & 0 & (\frac{5}{4})^3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 343 & 0 & 0 & 0 \\ 0 & 0 & 27 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{125}{64} \end{pmatrix}.$$

Upper and Lower Triangular Matrices

Triangular matrices are our next special type of matrix:

Definition. A square matrix is *upper triangular* if all of its below-diagonal entries are 0s. An upper triangular matrix has form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

A square matrix is *lower triangular* if all of its above-diagonal entries are 0s. A lower triangular matrix has form

$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

The first two matrices below are upper triangular, and the last is lower triangular:

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

In the example above, you should have noticed that the first and third matrices are just transposes. This is an illustration of the first part of the following theorem, which lists some important properties of triangular matrices:

Theorem 1.7.1. (a) The transpose of an upper triangular matrix is lower triangular, and vice versa.

(b) The product of two upper triangular matrices is upper triangular, and the product of two lower triangular matrices is lower triangular.

(c) A triangular matrix is invertible if and only if each of its diagonal entries is nonzero.

(d) The inverse of an invertible upper triangular matrix is lower triangular, and vice versa.

Symmetric Matrices

The matrix

$$A = \begin{pmatrix} 6 & -3 & 2 \\ -3 & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix}$$

has a special property, which you can discover by calculating $A^\top$:

$$A^\top = \begin{pmatrix} 6 & -3 & 2 \\ -3 & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

Notice that $A^\top$ and A are the same matrix, that is

$$A = A^\top.$$

As you may have suspected, we have a name for such special types of matrices:

Definition. A square matrix A is *symmetric* if $A = A^\top$. If A is symmetric, then $a_{ij} = a_{ji}$.

The following theorem lists some interesting properties of symmetric matrices:

Theorem 1.7.2. If A and B are symmetric $n \times n$ matrices, and k is any scalar, then:

(a) $A^\top$ is symmetric.

(b) $A + B$ and $A - B$ are symmetric.

(c) kA is symmetric.

Proof. Let's prove part (b) of the theorem. We'd like to show that, if A and B are symmetric, then so is $A + B$. Of course, if A and B are symmetric, then we know that

$$A = A^\top \text{ and } B = B^\top.$$

Now, to show that $A + B$ is symmetric, we need to be convinced that $(A + B)^\top = (A + B)$. Let's check that this is the case:

$$(A + B)^\top = A^\top + B^\top,$$

since the transpose of a sum is the sum of the transposes. Fortunately, combining this with the observation that $A = A^\top$ and $B = B^\top$ give us

$$\begin{aligned} (A + B)^\top &= A^\top + B^\top \\ &= A + B. \end{aligned}$$

Thus

$$(A + B)^\top = (A + B)$$

as we hoped, which means that $A + B$ is a symmetric matrix.

We should answer one final question in this section on symmetric matrices: If A and B are symmetric, is their product AB symmetric too? We can investigate this possibility with a few examples. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \qquad \text{and } C = \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix}.$$

Notice that each of A, B, and C is symmetric (i.e. $A = A^\top$, etc.). Per our question above, we should determine if products such as AB or AC are also symmetric. Let's calculate these two products:

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2+2 & 1+8 \\ 4+5 & 2+20 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 9 \\ 9 & 22 \end{pmatrix}. \end{aligned}$$

Notice that the product AB *is* indeed symmetric–i.e., $(AB)^\top = AB$. Let's check the next example, AC:

$$\begin{aligned} AC &= \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 2+2 & 1+14 \\ 4+5 & 2+35 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 15 \\ 9 & 37 \end{pmatrix}. \end{aligned}$$

Unfortunately, even though A and C are symmetric, their product AC is *not* a symmetric matrix.

We should try to pick apart what is going on here. We have seen that a product of two symmetric matrices *might* be symmetric itself, or it might not. What qualities of the matrices in question will control whether or not their product retains symmetry?

To answer this question, let's think about it in general, with a pair of matrices A and B (not necessarily the ones from the previous example) that we will assume are symmetric; $A = A^\top$, and $B = B^\top$. We want to understand the matrix AB–what conditions will make $(AB)^\top = AB$, so that AB is symmetric?

In Section 1.4, we learned a helpful fact about transposes of products of matrices:

$$(AB^\top) = B^\top A^\top. \qquad \text{(Theorem 1.4.8)}$$

Let's use this fact to help us out. We know that

$$(AB)^\top = B^\top A^\top.$$

Of course, if we also know that A and B are symmetric ($A = A^\top$!), then we can continue this line of reasoning:

$$\begin{aligned} (AB)^\top &= B^\top A^\top \\ &= BA. \end{aligned}$$

So on one hand, we are *certain* that

$$(AB)^\top = BA;$$

on the other hand, we know that, *if* AB is symmetric, then

$$(AB)^\top = AB.$$

We can conclude that, *if* AB happens to be symmetric, then

$$AB = (AB)^\top = BA.$$

In other words, if AB is symmetric, then A and B commute! If they do not commute, then AB cannot possibly be symmetric.

Our conclusion is precisely the statement of the following theorem:

Theorem 1.7.3. If A and B are symmetric matrices, then their product AB is symmetric *if and only if* A and B commute.

We finish off this section with a few more quick theorems about symmetric matrices:

Theorem 1.7.4. If A is both symmetric and invertible (so that A^{-1} exists), then A^{-1} is also symmetric.

Theorem. Given any $n \times m$ matrix A, the product $AA^\top$ of A with its transpose is an $n \times n$ symmetric matrix, and the product $A^\top A$ is an $m \times m$ symmetric matrix.

Proof. Exercise.