Elementary Matrices and an Inversion Algorithm

In Section 1.4, we introduced the idea of the *inverse* of an $n \times n$ matrix A, and discussed a formula for finding the inverse of a 2×2 matrix.

We would like to be able to find the inverse of matrices of sizes larger than 2×2; unfortunately, formulas for inverses become incredibly complicated as the size of the matrices in question increase. So instead of looking at *formulas for inverses* of large matrices in this section, we will instead introduce an *algorithm* that will allow us to find inverses.

In order to understand the algorithm, we must take a few moments to go back to the idea of elementary row operations.

Elementary Row Operations and Their Inverse Operations

In section 1.1, we introduced the elementary row operations:

- Multiply any row by a non-zero constant
- Interchange any pair of rows
- Add or subtract a constant multiple of one row from another.

One of the reasons that these operations are so helpful is that they are quite simple to "reverse" or "undo." For example, let A be the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}.$$

Let's apply the first row operation to A, and multiply the 2nd row of A by 4:

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ 4 \cdot 2 & 4 \cdot 2 & 4 \cdot 5 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 8 & 8 & 20 \\ -1 & 4 & 1 \end{pmatrix}.$$

If we wish to reverse this row operation (i.e., return to the original matrix A), all we need to do is *multiply the same row by* $\frac{1}{4}$:

$$\begin{pmatrix} 3 & 0 & 1 \\ 8 & 8 & 20 \\ -1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ \frac{1}{4}\cdot 8 & \frac{1}{4}\cdot 8 & \frac{1}{4}\cdot 20 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= A.$$

Key Point. The inverse operation to multiplying a row by a nonzero constant c is to multiply the same row by $\frac{1}{c}$.

Let's apply the second row operation to A, and interchange rows 1 and 2:

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}. \rightarrow \begin{pmatrix} 2 & 2 & 5 \\ 3 & 0 & 1 \\ -1 & 4 & 1 \end{pmatrix}$$

To reverse this operation, we simply need to *switch the same pair of rows*:

$$\begin{pmatrix} 2 & 2 & 5 \\ 3 & 0 & 1 \\ -1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= A.$$

Thus it is clear that

Key Point. The inverse operation to interchanging a pair of rows is to interchange the same two rows.

Finally, let's apply the third row operation to A, and add 2 times row 3 to row 2:

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ 2-2 & 2+8 & 5+2 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 10 & 7 \\ -1 & 4 & 1 \end{pmatrix}.$$

To reverse this operation, we should *add* -2 *times row* 3 *to row* 2:

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 10 & 7 \\ -1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0+2 & 10-8 & 7-2 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}$$

$$= A.$$

In general,

Key Point. The inverse operation to adding c times row i to row j is to add $-c$ times row i to row j.

If we can get from matrix A to matrix B via a sequence of elementary row operations, then the preceding discussion makes it clear that we can *go backwards* from B to A using the inverses of those operations. In other words, we can "recover" A if we know B and the necessary sequence of operations, so that there is a sense in which the two matrices are equivalent. This leads to the following definition:

Definition 1. Matrices A and B are *row equivalent* if one can be obtained from the other via a sequence of elementary row operations.

Using the previous example, matrices

$$\begin{pmatrix} -1 & 4 & 1 \\ 2 & 2 & 5 \\ 3 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}$$

are row equivalent since we can get from the first to the second by switching rows 1 and 3.

Elementary Matrices and Elementary Row Operations

It turns out that each of the elementary row operations can be accomplished via matrix multiplication using a special kind of matrix, defined below:

Definition 2. An *elementary matrix* is a matrix that can be obtained from I by using a *single* elementary row operation.

Example. Determine whether or not the following matrices are elementary; for each matrix that *is* elementary, find the row operation needed to obtain it from I:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. The matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
is elementary: we can obtain it from I by subtracting $\frac{1}{2}$ of row 1 from row 2:
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. The next matrix
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
is *not* elementary, since we are not allowed to multiply rows by 0.

3. The matrix
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
is elementary; we can switch rows 1 and 2 of I to obtain it:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. The final matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
is not elementary; it takes at least two elementary row operations (switching rows, multiplying a row by a constant) to obtain it from
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To understand where elementary matrices fit into our discussion of elementary row operations, let's go back to the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix}.$$

We saw that, upon interchanging rows 1 and 2 of A, we end up with the matrix

$$B = \begin{pmatrix} 2 & 2 & 5 \\ 3 & 0 & 1 \\ -1 & 4 & 1 \end{pmatrix}.$$

Now I claim that the elementary matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which we obtain by the *same* elementary row operation (switching rows 1 and 2), is a way to encode the row operation's action. To understand why, let's calculate the matrix product EA:

$$\begin{aligned} EA &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ -1 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0+2+0 & 0+2+0 & 0+5+0 \\ 3+0+0 & 0+0+0 & 1+0+0 \\ 0+0-1 & 0+0+4 & 0+0+1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 & 5 \\ 3 & 0 & 1 \\ -1 & 4 & 1 \end{pmatrix} \\ &= B. \end{aligned}$$

So there are two ways to obtain matrix B from A: we can either

- perform a row operation on A, or
- multiply A by the elementary matrix E that encodes the same operation.

The phenomenon observed above actually applies to *all* elementary matrices, as indicated by the following theorem:

Theorem 1.5.1. If the elementary matrix E results from performing a particular row operation on I_m, and A is an $m \times n$ matrix, then the product EA is the matrix that results from A by performing the *same* operation on A.

In essence, the theorem indicates that elementary matrices encode elementary row operations; applying an elementary row operation has the same effect as multiplying by the elementary matrix of the operation.

Inverses of Elementary Matrices

At the beginning of the section, we mentioned that every elementary row operation can be reversed. Since elementary row operations correspond to elementary matrices, the *reverse* of an operation (which is also an elementary row operation) should correspond to an elementary matrix, as well.

Theorem 1.5.2. Every elementary matrix E has an inverse, and E^{-1} is also elementary. In particular, E^{-1} is the elementary matrix encoding the inverse row operation from E.

For example, we have seen that the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

encodes the elementary row operation of adding $-\frac{1}{2}$ of row 1 to row 3. The inverse row operation–adding $\frac{1}{2}$ of row 1 to row 3–has elementary matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The theorem indicates that $F = E^{-1}$, which we can quickly check via multiplication:

$$\begin{aligned} EF &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2}+\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= I_4. \end{aligned}$$

Since $EF = I$, E and F are indeed inverses.

Checking For Invertibility

In the previous section, we mentioned that invertible matrices will play an important in our study of linear algebra; thus it will be helpful to be able to

1. determine whether or not a specific matrix has an inverse, and
2. find inverses when they exist.

The following theorem will provide us with several different ways to check a matrix for invertibility:

Theorem 1.5.3. Let A be an $n \times n$ matrix. Then the following are equivalent:

- A is invertible.
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A is a product of elementary matrices.

Remark. The phrase "the following are equivalent" means that either *all of the statements are true, or all of them are false.* So if, for example, I know that matrix A has reduced row echelon form

$$A \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then I can automatically conclude that A is not invertible; that the equation $A\mathbf{x} = \mathbf{0}$ has at least one nontrivial solution; and that A can not be factored as a product of elementary matrices.

Key Point. In section 1.4, we mentioned that the reduced row echelon form of a *square* matrix is always either:

1. the identity matrix I_n, or
2. a matrix with a row of 0s.

Taken in combination with Theorem 1.5.3 above, it is clear that there are only two possibilities for a square matrix A:

1. A is invertible, and its reduced row echelon form is I_n, or
2. A is singular, and its reduced row echelon form is a matrix with a row of 0s.

A Method for Calculating A^{-1}

Theorem 1.5.3 gives us a method for calculating the inverse of a nonsingular matrix A: if A is invertible, then its reduced row echelon form must be I_n.

In other words, there must be a sequence of elementary row operations which reduces A to I_n; as we saw earlier in this section, there must also be a sequence of elementary matrices E_1, E_2, ..., E_k corresponding to those row operations so that

$$E_k \dots E_2 E_1 A = I_n.$$

With a bit of matrix arithmetic, this statement can be rewritten as

$$A^{-1} = E_k \dots E_2 E_1 I_n.$$

In other words, *the sequence of elementary matrices which reduces A to I_n can also be used to calculate A^{-1}*.

This leads to a simple algorithm for calculating A^{-1}:

Use elementary row operations to reduce A to I_n; apply the same operations in the same order to I_n. If A is invertible, then the resulting matrix is A^{-1}.

Example. Find the inverse of

$$A = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

To help us keep track of all of the data here, let's start by augmenting A with I_4; as we apply elementary row operations to A, we will apply them to I as well:

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Let's start by creating a leading 1 in the first row:

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Next, we should create 0s below this leading 1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 1-1 & 3 & 1 & 0+1 & 0-\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1-1 & 2 & 0 & 0+1 & 0-\frac{1}{2} & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right).$$

Next, we want a leading 1 in row 2:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right).$$

Let's create 0 entries below the leading 1 from row 2:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2-2 & 0-\frac{2}{3} & 1-\frac{2}{3} & -\frac{1}{2}+\frac{1}{3} & 0-\frac{2}{3} & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right).$$

Next we need a leading 1 in row 3:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right),$$

followed by 0 entries below it:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3}+\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & 0-\frac{2}{9} & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & -\frac{2}{9} & 1 \end{array}\right)$$

and a leading 1 in the last row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} & -\frac{2}{9} & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right).$$

Finally, we need to create 0 entries above all of the leading 1s; starting with the last row, we have

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3}-\frac{1}{3} & -\frac{1}{6}+\frac{1}{6} & \frac{1}{3}+\frac{2}{3} & 0+\frac{2}{9} & 0-1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1+1 & \frac{1}{2}-\frac{1}{2} & 0-2 & 0-\frac{2}{3} & 0+3 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right).$$

Finally, we need to create 0 entries above the leading 1 from row 3:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 1 & \frac{2}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3}-\frac{1}{3} & 0 & 0 & 1 & \frac{2}{9}+\frac{1}{9} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & \frac{1}{3} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{array}\right).$$

Notice that we have now reduced A to the identity; what remains on the righthand side of the augmented matrix is A^{-1}. We conclude that

$$A^{-1} = \begin{pmatrix} 0 & -2 & -\frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & -1 \\ 0 & 0 & -\frac{1}{3} & 0 \\ -\frac{1}{2} & -2 & -\frac{2}{3} & 3 \end{pmatrix}.$$