Section 1.4

Properties of Matrix Operations

In section 1.3, we learned three operations on matrices: scalar multiplication, matrix addition, and matrix multiplication. These are indeed *new* operations to us, and so we need to discuss their properties in detail. To understand why we need to discuss their properties, consider the following example:

Example. Given

$$A = \begin{pmatrix} 2 & 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 2 \\ -3 & 10 \\ 1 & -2 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

compute ABC.

In order to compute ABC, we have to perform matrix multiplication *two times*. There's a bit of a problem here though–which product should we compute first? Potentially, we could:

1. multiply A and B to get the matrix AB, then calculate $(AB)C$, or
2. multiply B and C first to get the matrix BC, then calculate $A(BC)$.

This leads us to a problem–is repeated matrix multiplication "well-defined"? I.e., will we get the same answer regardless of how we go about performing the operations? As we have only just been introduced to the operation of matrix multiplication, we don't really have an answer to the question: $(AB)C$ might be the same matrix as $A(BC)$, or it might not. If they are different, then which one is the "right" way to make the calculation?

Fortunately for us, the list of properties below answers many such questions about the operations we learned in the previous section.

Properties of Scalar Multiplication, Matrix Addition, and Matrix Multiplication

Theorem 1.4.1. Assume that matrices A, B, and C have sizes that are amenable for performing the given operations, and let a and b be scalars. Then:

(a) $A + B = B + A$ (matrix addition is commutative)

(b) $(A + B) + C = A + (B + C)$ (matrix addition is associative)

(c) $(AB)C = A(BC)$ (matrix multiplication is associative)

(d) $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$ (matrix multiplication distributes over addition)

(h) $a(B + C) = aB + aC$ (scalar multiplication distributes over addition)

(j) $(a + b)C = aC + bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Most of the properties above are not particularly exciting; indeed, most of them are exactly what we would expect given our understanding of scalar addition and scalar multiplication. Regardless, we point out a few features of some of the properties:

(b) $(A + B) + C = A + (B + C)$: If we wish to add three matrices, we have to decide which two of the three to add first; this property simply says that the order doesn't matter–we'll end up with the same matrix either way.

(c) $(AB)C = A(BC)$: Similarly, if we wish to *multiply* three matrices, we can start by multiplying AB, and then C, or we can multiply A by the (already computed) product BC; either way, we'll end up with the same matrix. This idea is illustrated in the following example.

(m) $a(BC) = (aB)C = B(aC)$: In a sense, scalar multiplication commutes with matrix multiplication–scalars can move through products at will without affecting the outcome.

Remark. While many of the rules from Theorem 1.4.1 look familiar from real number arithmetic, you may have noticed that one familiar rule is missing–the commutativity of multiplication. Indeed, we have already seen an example that indicates that matrix multiplication is, in general, *not* commutative:

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -2 & 1 \\ 0 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -16 & 13 \\ 15 & -14 & 9 \\ -5 & 2 & -1 \end{pmatrix}, \text{ while } \begin{pmatrix} 5 & -2 & 1 \\ 0 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 16 \\ -15 & -8 \end{pmatrix}.$$

While there *are* occasions when $AB = BA$, for the most part multiplication is not commutative, and you are generally safe assuming that

$$AB \neq BA.$$

Example. Given

$$A = \begin{pmatrix} 2 & 0 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 4 & 2 \\ -3 & 10 \\ 1 & -2 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

verify that $(AB)C = A(BC)$.

Let's start by computing the matrix AB:

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -3 & 10 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 8+0-1 & 4+0+2 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 6 \end{pmatrix} \end{aligned}$$

Now that we know that

$$AB = \begin{pmatrix} 7 & 6 \end{pmatrix},$$

let's calculate $(AB)C$:

$$\begin{aligned} (AB)C &= \begin{pmatrix} 7 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0-6 & 7+0 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 7 \end{pmatrix}. \end{aligned}$$

Thus we have calculated that

$$(AB)C = \begin{pmatrix} -6 & 7 \end{pmatrix}.$$

On the other hand, we could start by finding the matrix BC:

$$\begin{aligned} BC &= \begin{pmatrix} 4 & 2 \\ -3 & 10 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0-2 & 4+0 \\ 0-10 & -3+0 \\ 0+2 & 1+0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 4 \\ -10 & -3 \\ 2 & 1 \end{pmatrix}. \end{aligned}$$

Now we can find $A(BC)$:

$$\begin{aligned} A(BC) &= \begin{pmatrix} 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ -10 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4+0-2 & 8+0-1 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 7 \end{pmatrix} \end{aligned}$$

Finally, we see that

$$(AB)C = \begin{pmatrix} -6 & 7 \end{pmatrix} = A(BC);$$

as promised by the theorem, the outcome is the same either way we make the computation.

Properties of the 0 Matrix

In section 1.3, we introduced the **0** matrix, all of whose entries are 0. While the **0** matrix is a different object from the *number* 0, it turns out that the **0** matrix behaves like the number 0 in many ways.

For example, we know that adding 0 to a number does not alter the number: $0 + a = a$, and we say that the number 0 acts as the *additive identity* for real numbers. Similarly, adding the **0** *matrix* to a matrix of the same size does not alter the matrix: $\mathbf{0} + A = A$, and we call **0** the *additive identity* for matrices.

The **0** matrix has several important properties that parallel the properties of the number 0; they are outlined below:

Theorem 1.4.2. Let A be an $m \times n$ matrix, and **0** be a zero matrix of size $m \times n$ (for (a), (c), and (e)) or size $k \times m$ (for (d)). Let c be any real number. Then:

(a) $\mathbf{0} + A = A + \mathbf{0} = A$

(c) $A + (-A) = \mathbf{0}$

(d) $\mathbf{0}A = \mathbf{0}$

(e) If $cA = \mathbf{0}$, then either $c = 0$ or $A = \mathbf{0}$.

Remark. As we discuss parallels between real number arithmetic and matrix arithmetic, it is important to point out times when such parallels fail. Our discussion of the **0** matrix leads directly to two such failures:

1. Cancelation: For real numbers, we know that if

$$ab = ac \text{ and } a \neq 0, \text{ then } b = c.$$

There is no such law for matrices; indeed it is perfectly possible to produce identical products from different matrices, as indicated by the following example. Let

$$A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix}.$$

A few quick calculations show that, even though $B \neq C$, $AB = AC$:

$$\begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix}.$$

2. 0 Products: If a and b are real numbers so that $ab = 0$, then we are guaranteed that at least one of a or b is 0. This is far from the case for matrices; setting

$$A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix},$$

it is easy to show that

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

even though neither of A nor B is **0**.

Properties of the Identity Matrix

We introduced another important matrix in section 1.3: the identity matrix I_n is the $n \times n$ (square) matrix whose main diagonal entries are all 1s, and all of whose other entries are 0s. For example, I_4 is given by

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Just as the **0** matrix parallels the number 0, I shares properties with the number 1 (which is referred to as the *multiplicative identity* for real numbers).

For example, the number 1 has the property that $1 \cdot a = a \cdot 1 = a$; in a sense it does not alter numbers multiplicatively. Similarly, I acts as a multiplicative identity for matrices:

Theorem. If A is an $m \times n$ matrix, then

$$I_m A = A \text{ and } AI_n = A.$$

The identity matrix turns out to play an important role in finding the reduced row echelon form of square matrices, as indicated by the following theorem:

Theorem 1.4.3. Let A be an $n \times n$ matrix. Then its reduced row echelon form is either:

1. I_n, or
2. a matrix with at least one row of 0s.

Inverses of Matrices

We have seen that many ideas from the world of numbers, such as addition and multiplication, have analogues in matrix theory. The tables below summarizes these ideas:

Real numbers		Matrices	
Operation	Outcome	Operation	Outcome
Real number addition	Real number	Matrix addition	Matrix
Real number multiplication	Real number	Matrix multiplication	Matrix

Real numbers		Matrices	
Object	Importance	Object	Importance
0 (as a number)	Additive identity	**0** (matrix)	Additive identity
1	Multiplicative identity	I	Multiplicative identity

It turns out that there are more analogues between the real numbers and matrices. For example, we know that *most* real numbers have a multiplicative inverse: for example, the multiplicative inverse of 5 is 1/5 since

$$5 \times \frac{1}{5} = 1.$$

The numbers 5 and 1/5 are multiplicative inverses since their product is the multiplicative identity 1. (Question: which number(s) have no multiplicative inverse?)

It turns out that there is an analogous idea for square matrices, as indicated by the following definition:

Definition 1. An $n \times n$ matrix A has a *multiplicative inverse* B if the product AB yields the multiplicative identity I_n, that is

$$AB = BA = I_n.$$

Remark. A few quick notes here:

- If B is the multiplicative inverse of the $n \times n$ matrix A, so that $AB = I_n$, then necessarily B must be $n \times n$ (check!).
- If $AB = I$, then it automatically follows that $BA = I$.

For example, the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

has inverse

$$B = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -3 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

You should verify via multiplication that

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

A matrix A that has an inverse is called *invertible* or *nonsingular*, and we refer to its inverse using the notation A^{-1}. Using the example above, we write

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -3 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

(Note: the notation A^{-1} does *not* refer to division in any way. Indeed, we do not even have a definition for matrix division, nor will we see one!)

Just as the *number* 0 has no multiplicative inverse, there are many matrices which do not have multiplicative inverses. For example, the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

has no inverse; it is easy to tell that this is the case, since the first row of B, which consists entirely of 0s, will force the first row of any product BC to consist entirely of 0s as well. A matrix which has no inverse is called *singular*.

One important question that we will need to answer is this: how can we determine whether a specific matrix A is singular or nonsingular? The question has a surprising answer, which we will study in this and several later sections.

One important and quite useful fact about inverses is given by the following theorem:

Theorem 1.4.4. If A, B, and C are $n \times n$ matrices so that

$$AB = I \text{ and } AC = I$$

(i.e. both B and C are inverse of A) then

$$B = C.$$

In other words, the inverse of a matrix is unique: if I find an inverse of A, I am guaranteed that it is the *only* one.

We can actually rephrase this theorem to get a version of the cancelation law:

Theorem 1.4.4. *(Cancelation Law Version)* If A, B, and C are $n \times n$ matrices so that A is invertible and

$$AB = AC,$$

then

$$B = C.$$

Key Point. It is important to note that Theorem 1.4.4 *only* works if A is invertible. We saw an example earlier where the cancelation law *did not* work: with

$$A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix},$$

it is easy to see that

$$AB = AC,$$

but of course $B \neq C$. The reason that cancelation fails is that A *is not* invertible (it has a row of 0s).

Finding Inverses of 2×2 Matrices

As indicated earlier, we would like to have a reliable method for

1. determining whether or not a specific matrix has an inverse, and
2. *finding* inverses when they exist.

In general, it is quite simple to tell if a matrix is invertible or not, but much more difficult to *find* that inverse. However, for the special case of 2×2 matrices, calculating inverses is actually quite easy. Thus we will focus on the 2×2 case for now, and leave larger matrices for a later discussion.

Theorem 1.4.5. (a) The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse *if and only if*

$$ad - bc \neq 0.$$

(b) If $ad - bc \neq 0$, then the inverse of A is the matrix

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof. Let's prove part (b) of the theorem: we would like to show that, if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To do so, we need to calculate AA^{-1}; if the theorem is true, then

$$AA^{-1} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let's check the theorem by multiplying:

$$\begin{aligned} AA^{-1} &= \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix} \\ &= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= \frac{1}{ad-bc}(ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I_2. \end{aligned}$$

Thus our claim is correct–since $AA^{-1} = I_2$, the theorem does indeed give the correct formula for A^{-1}.

Example. Determine if the following matrices are invertible, and find their inverses if possible:

1. $A = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix}$
2. $B = \begin{pmatrix} 6 & 4 \\ -4 & -\frac{8}{3} \end{pmatrix}$

1. To determine if
$$A = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix}$$
is invertible, we need to calculate the number $ad - bc$: with
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -4 & 10 \\ 2 & -3 \end{pmatrix}, \text{ we have } ad - bc = 12 - 20 = -8.$$
Since $ad - bc = -8 \neq 0$, this matrix is invertible; the formula from the theorem tells us that
$$A^{-1} = -\tfrac{1}{8} \begin{pmatrix} -3 & -10 \\ -2 & -4 \end{pmatrix}.$$
2. Unlike the previous matrix, it is clear that
$$B = \begin{pmatrix} 6 & 4 \\ -4 & -\frac{8}{3} \end{pmatrix}$$
is not invertible: in this case,
$$ad - bc = -16 + 16 = 0.$$

Properties of Inverses

We have yet to answer several important questions about the way that matrix inverses work. We might wish to know

- the inverse of a product AB;
- the inverse of a power A^r;
- or the inverse of a transpose $A^\top$.

We will answer these and several more related questions about inverses in the remainder of this section.

Inverses of Products

If we know that A and B are invertible $n \times n$ matrices with inverses A^{-1} and B^{-1}, respectively, it seems reasonable to guess that the product AB also has an inverse. Let's try to find a formula for the inverse $(AB)^{-1}$ of AB: we want a matrix X so that

$$(AB)X = I, \text{ or } A(BX) = I.$$

Since we are trying to determine a value for X, let's just leave it as a blank:

$$A(B\_\_\_) = I. \tag{1}$$

It seems reasonable to guess that A^{-1} and B^{-1} should play a roll in the desired formula. In fact, we can quickly eliminate the factor of B in (1) using B^{-1}:

$$A(BB^{-1}\_\_) = A(I\_\_) = A\_\_.$$

Of course, we still want

$$A(BB^{-1}\_\_) = A\_\_ = I.$$

We've certainly moved a bit closer to our goal–all we need to do now is fill in the blank in a way that eliminates the factor of A. Of course, we can do this easily: since

$$AA^{-1} = I,$$

we know that A^{-1} is the missing factor.

Let's look at what we've done: we filled in the blank in

$$A(B\_\_\_) = I$$

with $B^{-1}A^{-1}$. Indeed,

$$\begin{aligned} AB(B^{-1}A^{-1}) &= A(B(B^{-1}A^{-1})) \\ &= A((BB^{-1})A^{-1}) \\ &= A(IA^{-1}) \\ &= AA^{-1} \\ &= I, \end{aligned}$$

so it is clear that

Theorem 1.4.6. If A and B are both invertible matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Remark. We can actually generalize the theorem a bit: if each of A_1, A_2, ..., A_n is invertible, then so is the product

$$A_1A_2\ldots A_n,$$

and

$$(A_1A_2\ldots A_n)^{-1} = A_n^{-1}\ldots A_2^{-1}A_1^{-1}.$$

In other words, the inverse of a product is the product of the inverses in reverse order.

Inverses of Powers of Matrices

We can define integer powers of a square matrix in a natural way, analogous to the way that powers of real numbers are defined:

Definition. If r is an integer, $r > 0$, and A is an $n \times n$ matrix, then

$$A^r = \underbrace{A \cdot A \cdot \ldots \cdot A}_{r \text{ factors}}.$$

We define A^0 to be

$$A^0 = I_n.$$

It is quite easy to see that the usual rules for combining powers of real numbers also work for combining powers of matrices:

Theorem. If r and s are nonnegative real numbers, and A is square, then

- $A^r A^s = A^{r+s}$
- $(A^r)^s = A^{rs}$.

Of course, if A is invertible, with inverse A^{-1}, then we might wish to know the inverse of some power of A. As an example, let's find a formula for the inverse of A^3: we want

$$A^3 \underline{\hspace{2cm}} = I;$$

of course, if we replace the blank with three copies of A^{-1}, we should get the desired result. Let's check:

$$\begin{aligned}
A^3(A^{-1})^3 &= (A \cdot A \cdot A)(A^{-1} \cdot A^{-1} \cdot A^{-1}) \\
&= (A \cdot A) \cdot (A \cdot A^{-1}) \cdot (A^{-1} \cdot A^{-1}) \\
&= (A \cdot A) \cdot I \cdot (A^{-1} \cdot A^{-1}) \\
&= (A \cdot A) \cdot (A^{-1} \cdot A^{-1}) \\
&= A \cdot (A \cdot A^{-1}) \cdot A^{-1} \\
&= A \cdot I \cdot A^{-1} \\
&= A \cdot A^{-1} \\
&= I.
\end{aligned}$$

Thus we have verified that

$$(A^3)^{-1} = (A^{-1})^3;$$

in general,

Theorem 1.4.7. If A is an invertible $n \times n$ matrix, r is a nonnegative integer, and k is a nonzero scalar, then

- $(A^r)^{-1} = (A^{-1})^r$,
- $(A^{-1})^{-1} = A$, and
- $(kA)^{-1} = k^{-1}A^{-1}$.

Instead of using the clumsy notation $(A^{-1})^r$, we use A^{-r}.

Key Point. The notation A^{-r} indicates the matrix that is the inverse of A^r.

Inverses (and Other Properties) of Transposes

Recall that the transpose of a matrix A is the matrix $A^\top$ that we build by switching the rows and columns of A. For example, if

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}, \text{ then } A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 5 \end{pmatrix}.$$

If A is a square invertible matrix, then we should be able to find a formula for the inverse $(A^\top)^{-1}$ of $A^\top$; it turns out that

Theorem 1.4.9. If the $n \times n$ matrix A is invertible, then so is $A^\top$, and

$$(A^\top)^{-1} = (A^{-1})^\top.$$

In other words, to find the inverse of the transpose of A, just calculate the inverse of A and take *its* transpose!

A few more helpful properties of transposes are given by the following theorem:

Theorem 1.4.8. Assume that A and B are matrices of the appropriate sizes to perform the indicated operations, and let k be any scalar. Then

(a) $(A^\top)^\top = A$

(b) $(A + B)^\top = A^\top + B^\top$

(d) $(kA)^\top = kA^\top$

(e) $(AB)^\top = B^\top A^\top$.