
Solving Linear First-Order Equations

In this section, we will learn a technique that will allow us to solve *linear first-order* differential equations. Recall that a *first-order* differential equation is one in which the highest derivative that appears is the first. For example,

$$xy' + y = 3x$$

is a first order equation, but

$$xy'' + y = 3x$$

is not—it's an example of a second-order differential equation.

In addition, when we say that an equation is *linear*, we mean that there are no ys or $y's$ raised to a power higher than 1. A typical linear first-order differential equation has form

$$y' + P(x)y = Q(x).$$

For example,

$$y' - 2y = 4 - x$$

is a linear first-order differential equation, and the first example that we will solve.

We need to make one minor note here: in 1.4, we learned a special technique for solving separable equations, whereas in 1.5 we will learn how to solve linear first-order equations. However, every separable equation is *also* a linear first-order equation; so the problems we learned to solve in 1.4 can also be solved using the techniques from this section. The caveat is that the method we learned in 1.4 is relatively simple, and so I recommend reverting to this method if you recognize that a linear first-order equation is also separable.

This leads to one more remark: while every separable equation is also a linear first-order equation, *not* every linear first-order equation is separable. The first problem we will work through is a good example:

$$y' = 2y + 4 - x$$

is not separable since $2y + 4 - x$ can't be factored as a product of a function of y and a function of x . Thus we must solve it using the technique introduced below.

The solution method that we are about to learn is a bit tricky to understand and may seem unmotivated, but as it is easy to use and will solve *any* linear first-order differential equation (assuming that we can evaluate certain integrals), it is an excellent technique to have in our mathematical toolbox.

The general idea behind the method is to multiply the entire equation through by a carefully chosen function, known as the *integrating factor* (believe it or not, this will significantly simplify the problem). The integrating factor for the linear first-order equation

$$y' + P(x)y = Q(x)$$

is

$$\rho(x) = e^{\int P(x) dx},$$

so we will rewrite the original equation as

$$\rho(x)y' + \rho(x)P(x)y = \rho(x)Q(x), \text{ or}$$

$$e^{\int P(x) dx} y' + e^{\int P(x) dx} P(x)y = e^{\int P(x) dx} Q(x).$$

This rewritten equation looks like a bit of a disaster, but its purpose will become clearer after a brief digression. I'd like to take the derivative with respect to x of

$$\rho(x)y = e^{\int P(x) dx} y;$$

since y is a function of x , we'll use the chain rule:

$$\begin{aligned} \frac{d}{dx} \rho(x)y &= \rho(x)y' + \rho'(x)y \\ &= e^{\int P(x) dx} y' + \left(\frac{d}{dx} e^{\int P(x) dx} \right) y \\ &= e^{\int P(x) dx} y' + P(x)e^{\int P(x) dx} y \\ &= \rho(x)y' + \rho(x)P(x)y. \end{aligned}$$

But this last line is something we have already seen, in the equation

$$\rho(x)y' + \rho(x)P(x)y = \rho(x)Q(x).$$

In other words, $\rho(x)y' + \rho(x)P(x)y$ is a derivative, in particular, the derivative of the function $\rho(x)y$. We know how to integrate derivatives:

$$\int \rho(x)y' + \rho(x)P(x)y dx = \rho(x)y + C.$$

So going back to the equation

$$\rho(x)y' + \rho(x)P(x)y = \rho(x)Q(x),$$

we already know how to integrate the left-hand side; since the right-hand side is a function of x , we should be able to integrate it as well to solve the problem.

Example. Solve the differential equation $y' - 2y = 4 - x$.

Notice that this equation is indeed a linear first order equation; we can match it up with the form $y' + P(x)y = Q(x)$ by thinking of $P(x) = -2$ and $Q(x) = 4 - x$.

In this example, our integrating factor is

$$\rho(x) = e^{\int P(x) dx} = e^{\int -2 dx}.$$

Let's simplify things a bit by evaluating the integral in the exponent: since

$$\int -2 dx = -2x + C,$$

the integrating factor is

$$\rho(x) = e^{-2x}$$

(we'll ignore the constant of integration for now).

Multiplying the original equation through by the integrating factor gives us the new equation

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}(4 - x),$$

which is equivalent to

$$\int e^{-2x}y' - 2e^{-2x}y \, dx = \int e^{-2x}(4 - x) \, dx.$$

But we already know that $e^{-2x}y' - 2e^{-2x}y$ is the derivative of $\rho(x)y = e^{-2x}y$ (you should verify this!), so that

$$\int e^{-2x}y' - 2e^{-2x}y \, dx = \int e^{-2x}(4 - x) \, dx$$

becomes

$$e^{-2x}y = \int e^{-2x}(4 - x) \, dx.$$

Solving the problem now amounts to evaluating the integral on the right.

Since the integrand $e^{-2x}(4 - x)$ is a product of two functions of x , we should probably try integration by parts. Set

$$\begin{aligned} u &= 4 - x & dv &= e^{-2x} \, dx \\ du &= - \, dx & v &= -\frac{1}{2}e^{-2x} \end{aligned}$$

so that

$$\begin{aligned} \int e^{-2x}(4 - x) \, dx &= uv - \int v \, du \\ &= (4 - x)\left(-\frac{1}{2}e^{-2x}\right) - \int \left(-\frac{1}{2}e^{-2x}\right)(- \, dx) \\ &= (4 - x)\left(-\frac{1}{2}e^{-2x}\right) - \int \left(\frac{1}{2}e^{-2x}\right) \, dx \\ &= (4 - x)\left(-\frac{1}{2}e^{-2x}\right) + \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Now we can assemble all of our information: we know that

$$\begin{aligned} \int e^{-2x}y' - 2e^{-2x}y \, dx &= \int e^{-2x}(4 - x) \, dx, \\ \int e^{-2x}y' - 2e^{-2x}y \, dx &= e^{-2x}y, \text{ and} \\ \int e^{-2x}(4 - x) \, dx &= (4 - x)\left(-\frac{1}{2}e^{-2x}\right) + \frac{1}{4}e^{-2x} + C. \end{aligned}$$

So we have

$$e^{-2x}y = (4 - x)\left(-\frac{1}{2}e^{-2x}\right) + \frac{1}{4}e^{-2x} + C,$$

which simplifies to

$$y = -\frac{1}{2}(4 - x) + \frac{1}{4} + Ce^{2x},$$

or

$$y = \frac{x}{2} - \frac{7}{4} + Ce^{2x}.$$

Example. Solve the initial value problem $xy' + 2y = 4x^2$, $y(1) = 2$.

We can force this equation to be in linear first-order form by doing a bit of rewriting: multiplying both sides of the equation by x^{-1} yields

$$y' + \frac{2y}{x} = 4x,$$

with

$$P(x) = \frac{2}{x} \text{ and } Q(x) = 4x.$$

The integrating factor is

$$\begin{aligned} e^{\int P(x) \, dx} &= e^{\int \frac{2}{x} \, dx} \\ &= e^{2 \ln |x|} \\ &= e^{\ln x^2} \\ &= x^2. \end{aligned}$$

Next we multiply the original equation through by the integrating factor, giving us

$$x^2y' + 2xy = 4x^3;$$

we'd like to integrate each side of this equation, i.e.

$$\int x^2y' + 2xy \, dx = \int 4x^3 \, dx.$$

Since we already know that

$$\frac{d}{dx}x^2y = x^2y' + 2xy,$$

we have

$$\int x^2y' + 2xy \, dx = x^2y.$$

Again, solving the problem comes down to evaluating the integral on the right:

$$\int 4x^3 \, dx = x^4 + C,$$

so that

$$x^2y = x^4 + C.$$

Thus our solution is

$$y = x^2 + \frac{C}{x^2}.$$

We can determine the value for C using the initial condition $y(1) = 2$: we have

$$\begin{aligned} 2 &= y(1) \\ &= 1 + \frac{C}{1} \end{aligned}$$

so that $C = 1$. Our final solution is

$$y = x^2 + \frac{1}{x^2}.$$

Example (Application). A 500 gallon continuously stirred tank reactor (CSTR) contains 200 gallons of salt-water; 100 pounds of salt is dissolved in the solution. At time $t = 0$, a solution containing 1 pound of salt per gallon begins to flow into the CSTR at a rate of 3 gallons per minute, and the well-stirred solution begins to flow out at a rate of 2 gallons per minute. How many pounds of salt does the solution contain at the point in time at which the CSTR is about to overflow?

As with the example in section 1.4, we would like to come up with a function $A(t)$ whose value at time t is the number of pounds of salt in the tank. The change in salt over time (A') is only affected by the amount of salt added (A_i) and the amount of salt drained (A_o), so that

$$A' = A_i - A_o.$$

Let's determine A_i and A_o .

The amount of salt being added to the tank is constant; 3 gallons of solution per minute is entering the tank, 1 pound of which is salt. So the amount of salt entering the tank at time t is

$$A_i = 3 \cdot 1 = 3 \text{ pounds per minute.}$$

On the other hand, the amount of salt being drained will change over time. In addition, the number of gallons of solution in the tank is also changing. Since solution is entering at 3 gallons per minute but exiting at 2 gallons per minute, the amount of solution is increasing by 1 gallon per minute; so at time t , there are $200 + t$ gallons of solution in the tank. So the portion of solution that is salt is

$$\frac{\text{current amount of salt in the tank}}{\text{number of gallons of solution in the tank}} = \frac{A}{200 + t}.$$

Since the solution is draining at 2 gallons per minute, we have

$$A_o = 2 \cdot \frac{A}{200 + 2} = \frac{2A}{200 + t} \text{ pounds per minute.}$$

Thus our differential equation is given by

$$A' = 3 - \frac{2A}{200 + t}.$$

We can see that this is a linear first-order equation by rewriting it as

$$A' + \frac{2A}{200 + t} = 3,$$

with

$$P(t) = \frac{2}{200+t} \text{ and } Q(t) = 3.$$

Using the technique we learned above, we'll need to multiply this equation through by the integrating factor

$$\begin{aligned} e^{\int P(t) dt} &= e^{\int \frac{2}{200+t} dt} \\ &= e^{2 \ln |200+t|} \\ &= e^{\ln(200+t)^2} \\ &= (200+t)^2. \end{aligned}$$

Next we multiply the original equation through by the integrating factor:

$$A' + \frac{2A}{200+t} = 3 \text{ becomes}$$

$$(200+t)^2 A' + 2(200+t)A = 3(200+t)^2,$$

and we will solve by integrating both sides of the equation,

$$\int (200+t)^2 A' + 2(200+t)A dt = \int 3(200+t)^2 dt.$$

We know that

$$\int (200+t)^2 A' + 2(200+t)A dt = (200+t)^2 A,$$

so we simply need to evaluate the right-hand integral:

$$\begin{aligned} \int 3(200+t)^2 dt &= 3 \int (200^2 + 400t + t^2) dt \\ &= 3(200^2 t + 200t^2 + \frac{t^3}{3}) + C \\ &= 120,000t + 600t^2 + t^3 + C. \end{aligned}$$

So

$$\int (200+t)^2 A' + 2(200+t)A dt = \int 3(200+t)^2 dt$$

becomes

$$(200+t)^2 A = 120,000t + 600t^2 + t^3 + C;$$

solving for A , we have

$$A = \frac{120,000t + 600t^2 + t^3 + C}{(200+t)^2}.$$

Since there were 100 pounds of salt in the solution at time $t = 0$ (i.e. $A(0) = 100$), the value for C is

$$C = 100 \cdot 200^2 = 4,000,000.$$

Thus the function describing the amount of salt in the solution at time $t = 0$ is

$$A = \frac{120,000t + 600t^2 + t^3 + 4,000,000}{(200 + t)^2}.$$

We originally wanted to know the amount of salt at the point in time where the tank is about to overflow, i.e. when there are 500 gallons of solution in the CSTR. Since the amount of solution in the tank at time t is $200 + t$, the CSTR is about to overflow after $t = 300$ minutes. The amount of salt in the tank at that time is

$$A(300) = \frac{120,000(300) + 600(300)^2 + (300)^3 + 4,000,000}{(200 + 300)^2} = 484 \text{ pounds.}$$