Impulse Functions

In many application problems, an external force $f(t)$ is applied over a very short period of time. For example, if a mass in a spring and dashpot system is struck by a hammer, the application of the force to the mass happens almost instantaneously.

In such physical contexts, it is acceptable to simplify the mathematical model by assuming that the application of the force *is* actually instantaneous; in other words, we think of the force as a function that takes on a nonzero value at only one point t in time.

Mathematicians have constructed a reasonable way to model such a phenomenon. We create an object $\delta(t-a)$ known as the *Dirac delta function.* The name is rather misleading, as δ is not actually a function; instead, we will think of it is an ad-hoc tool for modeling situations in which a force $f(t)$ is applied over an extremely short period of time.

When a force $f(t)$ is applied to a mass in a system, we recall that Newton's law says that

$$f(t) = ma = m\frac{\mathrm{d}}{\mathrm{d}t}v,$$

where m is the object's mass, a its acceleration, and v its velocity. If the force is applied from $t = a$ to $t = b$, we define

$$\int_a^b f(t)\ \mathrm{d}t$$

to be the *impulse* of the force on the interval $a \le t \le b$.

Of course, using the formula above from Newton's law, we can rewrite the impulse as

$$\begin{aligned}\int_a^b f(t)\ \mathrm{d}t &= \int_a^b m\frac{\mathrm{d}}{\mathrm{d}t}v\ \mathrm{d}t \\ &= mv\big|_a^b \\ &= m(v(b)-v(a)).\end{aligned}$$

In other words, the impulse of a function $f(t)$ on the interval $a \le t \le b$ is just the *change in momentum* of the mass in the system, so that we can think of the impulse as a measurement of the strength of the force.

In order to describe an "instantaneous" force at time $t = a$, we wish to have a forcing function $\delta_a(t)$ whose only nonzero value occurs at $t = a$, and whose impulse at $t = a$ is 1; mathematically, this means that

$$\lim_{b\to a}\int_a^b \delta_a(t)\ \mathrm{d}t = 1.$$

Thinking of the integral above as a description of the area under $\delta_a(t)$, we see that the statement above indicates that the "area" under $\delta_a(t)$ from a to b must approach ∞ as $b \to a$; to satisfy all of these requirements, $\delta_a(t)$ must have form

$$\delta_a(t) = \begin{cases}\infty & \text{if } t = a \\ 0 & \text{if } t \ne a\end{cases},$$

and the property that

$$\int_0^\infty \delta_a(t)\ \mathrm{d}t = 1.$$

So for any other function $g(t)$, it can be shown that

$$\int_0^\infty g(t)\delta_a(t)\ \mathrm{d}t = g(a);$$

in particular, we can "calculate" the Laplace transform of $\delta_a(t)$:

$$\mathcal{L}\{\delta_a(t)\} = \int_0^\infty e^{-st}\delta_a(t)\ \mathrm{d}t = e^{-as}.$$

As with the unit step function $u(t-a)$, we rewrite $\delta_a(t)$ as $\delta(t-a)$; then using the ideas above as motivation, we define the Dirac delta function $\delta(t-a)$ to be the object with the following properties:

$$\begin{aligned} \int_0^\infty \delta(t-a)\ \mathrm{d}t &= 1 \\ \delta(t-a) &= \begin{cases} \infty & \text{if } t = a \\ 0 & t \neq a \end{cases} \\ \mathcal{L}\{\delta(t-a)\} &= e^{-as} \end{aligned}$$

In particular, we interpret $\delta(t-a)$ as a description of a force applied (nearly) instantaneously at time $t = a$ with impulse 1.

Example. A 1 kg mass in a spring and dashpot system is attached to a spring with spring constant $k = 13$ and a dashpot with damping constant $c = 6$. At time $t = 0$, the mass is released from its equilibrium position with an initial velocity of 1 m/s to the left. At time $t = \pi$ and $t = 2\pi$, the mass is struck by a hammer with impulse $p = 3$. Find the equation $x(t)$ modeling the position of the mass.

Starting with the equation $mx'' + cx' + kx = f(t)$, we have $m = 1$, $c = 6$, and $k = 13$; we know that $x(0) = 0$ and $x'(0) = -1$. In addition, we can approximate the external force $f(t)$ by using the unit impulse functions $\delta(t-\pi)$ and $\delta(t-2\pi)$; of course, since the impulse in both cases is $p = 3$, we think of $f(t)$ as

$$f(t) = 3\delta(t-\pi) + 3\delta(t-2\pi).$$

Thus we need to solve the initial value problem

$$x'' + 6x' + 13x = 3\delta(t-\pi) + 3\delta(t-2\pi),\ x(0) = 0,\ x'(0) = -1.$$

Applying the Laplace transform to the entire equation, we have

$$\begin{aligned} s^2X(s) - sx(0) - x'(0) + 6sX(s) - 6x(0) + 13X(s) &= 3e^{-\pi s} + 3e^{-2\pi s} \\ s^2X(s) + 1 + 6sX(s) + 13X(s) &= 3e^{-\pi s} + 3e^{-2\pi s} \\ s^2X(s) + 6sX(s) + 13X(s) &= 3e^{-\pi s} + 3e^{-2\pi s} - 1 \\ X(s)(s^2 + 6s + 13) &= 3e^{-\pi s} + 3e^{-2\pi s} - 1 \\ X(s) &= \frac{3e^{-\pi s}}{s^2 + 6s + 13} + \frac{3e^{-2\pi s}}{s^2 + 6s + 13} - \frac{1}{s^2 + 6s + 13} \\ X(s) &= \frac{3e^{-\pi s}}{(s+3)^2 + 4} + \frac{3e^{-2\pi s}}{(s+3)^2 + 4} - \frac{1}{(s+3)^2 + 4}. \end{aligned}$$

Next, we need to apply the inverse transform to return to an equation for $x(t)$; we'll consider each term separately. Beginning with

$$\frac{3e^{-\pi s}}{(s+3)^2+4},$$

we note that we can think of this function as

$$\frac{3}{2}e^{-\pi s}\left(\frac{2}{(s+3)^2+4}\right),$$

whose inverse transform we can find using Theorem 1 from section 7.5:

$$\text{If } \mathcal{L}\{g(t)\} = G(s), \text{ then } \mathcal{L}^{-1}\{e^{-as}G(s)\} = u(t-a)g(t-a).$$

We simply need to find the inverse transform of

$$G(s) = \frac{2}{(s+3)^2+4};$$

we recognize this as a translation of

$$\frac{2}{s^2+4},$$

whose inverse transform is

$$\sin 2t;$$

thus the inverse transform of $G(s)$ is

$$g(t) = e^{-3t}\sin 2t.$$

Thus the inverse Laplace transform of our original fraction is

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{(s+3)^2+4}\right\} &= \frac{3}{2}u(t-\pi)g(t-\pi)\\ &= \frac{3}{2}u(t-\pi)e^{-3(t-\pi)}\sin(2(t-\pi)).\end{aligned}$$

Next, we need to find the inverse transform of

$$\frac{3e^{-2\pi s}}{(s+3)^2+4} = \frac{3}{2}e^{-2\pi s}\frac{2}{(s+3)^2+4};$$

using the same ideas, we think of $G(s)$ as

$$G(s) = \frac{2}{(s+3)^2+4},$$

whose inverse transform is

$$g(t) = e^{-3t}\sin 2t;$$

thus the inverse transform of

$$\frac{3e^{-2\pi s}}{(s+3)^2+4}$$

is

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3e^{-2\pi s}}{(s+3)^2+4}\right\} &= \frac{3}{2}u(t-2\pi)g(t-2\pi)\\ &= \frac{3}{2}u(t-2\pi)e^{-3(t-2\pi)}\sin(2(t-2\pi)).\end{aligned}$$

Finally, the inverse transform of

$$\frac{1}{(s+3)^2+4} = \frac{1}{2}\frac{2}{(s+3)^2+4}$$

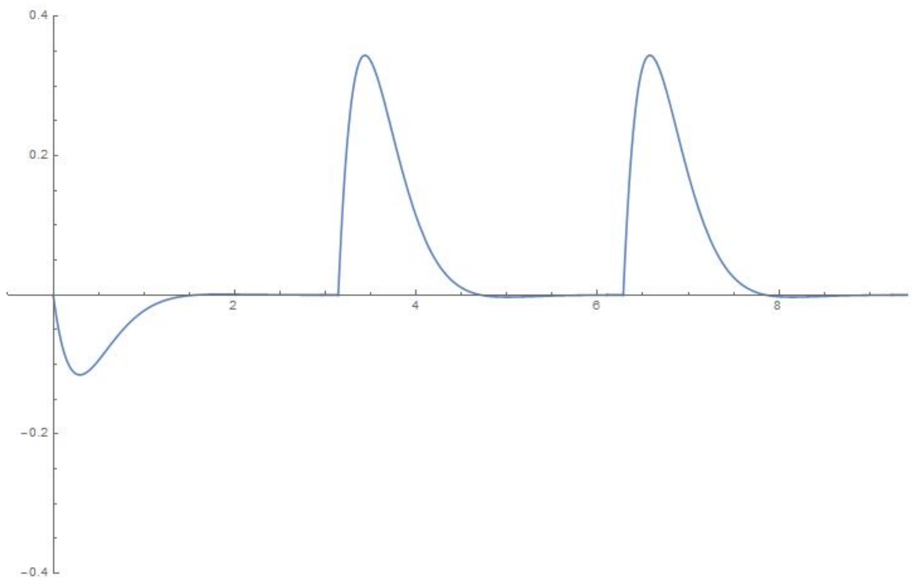
is

$$\frac{1}{2}e^{-3t}\sin 2t.$$

Thus we have

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{(s+3)^2+4} + \frac{3e^{-2\pi s}}{(s+3)^2+4} - \frac{1}{(s+3)^2+4}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{(s+3)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{3e^{-2\pi s}}{(s+3)^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+4}\right\}\\ &= \frac{3}{2}u(t-\pi)e^{-3(t-\pi)}\sin(2(t-\pi)) + \frac{3}{2}u(t-2\pi)e^{-3(t-2\pi)}\sin(2(t-2\pi)) - \frac{1}{2}e^{-3t}\sin 2t.\end{aligned}$$

The equation for $x(t)$ is graphed below:



Example. A 1 kg mass is attached to a spring with $k = 1$. Initially, the mass is at rest at equilibrium. At $t = 0$, the mass is struck with a hammer with impulse $p = 1$; the blow is repeated at each integer multiple of π. Find the function $x(t)$ modeling the mass's position.

Beginning with the equation $mx'' + cx' + kx = f(t)$, $x(0) = x'(0) = 0$, we have $m = 1$, $c = 0$, and $k = 1$. We can think of the external force $f(t)$ as an infinite sum of unit impulse functions:

$$\begin{aligned} f(t) &= \delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \ldots + \delta(2-n\pi) + \ldots \\ &= \sum_{n=0}^{\infty} \delta(t-n\pi), \end{aligned}$$

whose Laplace transform is

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{\delta(t)\} + \mathcal{L}\{\delta(t-\pi)\} + \mathcal{L}\{\delta(t-2\pi)\} + \ldots + \mathcal{L}\{\delta(t-n\pi)\} + \ldots \\ &= 1 + e^{-\pi s} + e^{-2\pi s} + \ldots + e^{-n\pi s} + \ldots \\ &= \sum_{n=0}^{\infty} e^{-n\pi s}. \end{aligned}$$

Thus our initial value problem is

$$x'' + x = \sum_{n=0}^{\infty} \delta(t-n\pi), \;\; x(0) = x'(0) = 0.$$

Applying the Laplace transform to each side of the equation, we have

$$\begin{aligned} s^2X(s) - sx(0) - x'(0) + X(s) &= \sum_{n=0}^{\infty} e^{-n\pi s} \\ s^2X(s) + X(s) &= \sum_{n=0}^{\infty} e^{-n\pi s} \\ X(s) &= \sum_{n=0}^{\infty} \frac{e^{-n\pi s}}{s^2+1}. \end{aligned}$$

Since the inverse transform of

$$G(s) = \frac{e^{-n\pi s}}{s^2+1} \text{ is } g(t) = u(t-n\pi)\sin(t-n\pi),$$

the formula for $x(t)$ is

$$x(t) = \sum_{n=0}^{\infty} u(t-n\pi)\sin(t-n\pi).$$

Of course, we can use an identity to rewrite $\sin(t-n\pi)$:

$$\begin{aligned} \sin(t-n\pi) &= \sin t \cos n\pi - \cos t \sin n\pi \\ &= \sin t \cos n\pi, \\ &= \begin{cases} \sin t & n \text{ even} \\ -\sin t & n \text{ odd.} \end{cases} \end{aligned}$$

Rewriting the formula for $x(t)$, we have

$$\begin{aligned} x(t) &= \begin{cases} \sin t & t < \pi \\ \sin t - \sin t & \pi \le t < 2\pi \\ \sin t - \sin t + \sin t & 2\pi \le t < 3\pi \\ \vdots \end{cases} \\ &= \begin{cases} \sin t & t < \pi \\ 0 & \pi \le t < 2\pi \\ \sin t & 2\pi \le t < 3\pi \\ \vdots \end{cases} \end{aligned}$$

or

$$x(t) = \begin{cases} \sin t & 2k\pi \le t < (2k+1)\pi \\ 0 & (2k+1)\pi \le t < (2k+2)\pi. \end{cases}$$

The function $x(t)$ is graphed below: