Section 3.1

An Introduction to Higher-Order Differential Equations

Up to this point in the class, we have only specifically studied solution techniques for first-order differential equations, i.e. equations whose highest derivative is the first. For example,

\[ y' - 2y = 4 - x \]

is a first-order differential equation; we solved it in section 1.5. We do not yet have the tools to solve an equation like

\[ y'' - 2y = 4 - x; \]

this equation is second-order since its highest derivative is the second.

In this chapter, we will study solution techniques for differential equations of order 2, like

\[ y'' - 2y = 4 - x, \]

as well as higher-order equations. However, before we proceed, it would be helpful to spend a few moments refreshing our memories on the information we know about first-order differential equations; the information we will gather about higher-order equations, while complementary to what we have learned about first-order equations, can at first glance seem contradictory, so it will be helpful to solidify our understanding of first-order equations before proceeding.

Summary of First-Order Differential Equations

The equation

\[ \frac{dy}{dx} = F(x, y) \]

is a first-order differential equation; the left-hand side is the first derivative \( y' \), while the right-hand side is a function of \( x \)s and \( y \)s, and no higher-order derivatives show up.

We have seen three specific forms in which a first-order differential equation might appear, and an appropriate solution method for each form. These forms are summarized below:

<table>
<thead>
<tr>
<th>Name</th>
<th>Form</th>
<th>Solution method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrable</td>
<td>( y' = f(x) )</td>
<td>integrate</td>
</tr>
<tr>
<td>Separable</td>
<td>( y' = f(x)g(y) )</td>
<td>separate variables, integrate</td>
</tr>
<tr>
<td>Linear first-order</td>
<td>( y' + P(x)y = Q(x) )</td>
<td>multiply by integrating factor, integrate</td>
</tr>
</tbody>
</table>

Given the equation

\[ \frac{dy}{dx} = F(x, y), \]

we attempt to find a general solution or family of solutions to the equation; in particular, this general solution will include a constant \( C \). For example, the curves graphed below are all in the family of solutions of the form \( y = Ce^{3x} \) to the differential equation \( y' = 3y \):
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Some solution curves in the family of solutions for differential equation $3y = y'$.

If we specify an initial condition to go along with the equation, then we are in effect choosing a specific value for the constant $C$; this amounts to choosing a specific curve from the family of solutions. In the example above, the initial condition $y(0) = 2$ forces us to choose $C = 2$, so that we end up with the particular solution highlighted below:

Solution curve to the initial value problem $3y = y', \ y(0) = 2$.

Now what if there were some other curve $y(x)$ that satisfied the equation $3y = y'$ and passed through the point $(0, 2)$, just like $y = 2e^{3x}$?

Fortunately, we saw that we don’t have to worry about this. If the initial value problem

$$
\frac{dy}{dx} = F(x, y), \ y(a) = b,
$$

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is such that $F(x, y)$ and $D_y F(x, y)$ are continuous near the point $(a, b)$, then there is a unique curve $y$ satisfying the equation and passing through $(a, b)$.

In our example, the functions $F(x, y)$ and $D_y F(x, y)$ are given by

$$F(x, y) = 3y \quad \text{and} \quad D_y F(x, y) = 3.$$ 

Clearly both of these functions are continuous on the entire $xy$ plane, and in particular are continuous near the point $(0, 2)$ (we could choose the box $-1 \leq x \leq 1, 0 \leq y \leq 4$, if we like). Thus we are assured that $y = 2e^{3x}$ is the only solution to $3y = y'$ passing through $(0, 2)$.

In summary, there are a few important points to notice about first-order differential equations:

(1) first-order equations can come in several different forms; it is essential to be able to recognize the form and to use the correct solution technique when attempting to find a solution to the equation;

(2) the general solution of a first order equation is a function which includes one arbitrary constant $C$; and

(3) if our first-order equation is "well-behaved", we need only specify one piece of data (the initial condition) to specify $C$, which amounts to choosing the unique particular solution curve from among all of the curves embodied by the general solution.

As we move on to study higher-order equations, all of the ideas above will have analogues; in particular, we will be asking the same questions about uniqueness and existence of solutions, about how many constants a general solution should contain, and about how much data we must specify to guarantee that a higher-order equation has a unique solution. Of course, the ideas will be a bit more complicated, but recalling the first-order ideas can help greatly with understanding the higher-order concepts.

Introduction to Higher-order Differential Equations

In this chapter, we will study higher-order differential equations. Below are a few examples:

$$xy'' - 3y' + 2\frac{y}{x} = 4\sin 3x \quad \text{second-order differential equation}$$

$$3y''' + 2y'' - 4y' + 8y = 3e^{-x} + 5x^2 \quad \text{third-order differential equation}$$

$$y^{(4)} + xy = 0 \quad \text{fourth-order differential equation}$$

In reference to point (1) that we made above regarding first-order equations, it should be clear that higher-order equations may come in many different forms as well, and it is vital to be able to recognize the forms in order to apply a correct solution method. Below we classify characteristics of higher-order equations that will help us determine how to solve them.

Characteristics of Higher-Order Equations

A differential equation is linear if
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(a) the only powers to which $y$ and each of its derivatives are raised are 0 and 1, and

(b) the coefficients of $y$ and each of its derivatives are functions of $x$ (and not functions of $y$).

In particular, the word linear only applies to the variable $y$ and its derivatives; $x$ may show up in any form we like.

Every $n$th order linear differential equation is of the form

$$A_n(x)y^{(n)} + \ldots + A_2(x)y'' + A_1(x)y' + A_0(x)y = F(x);$$

assuming that $A_n(x)$ is non-zero on the interval over which the equation is defined, we may divide through by $A_n(x)$, rewriting the equation above as

$$y^{(n)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x).$$

**Example.** The differential equation

$$(y'')^2 + y = 0$$

is nonlinear, because $y''$ is raised to the second power, violating condition (a). The equation

$$y'' \sin y = x$$

is nonlinear because $y''$ is multiplied by a function of $y$, violating condition (b).

**Example.** On the other hand,

$$2y''' + 3y' - \frac{y}{2} = x^3 \text{ and } xy'' - 3x^2 y' - \frac{y}{x} = 0$$

are both linear equations. None of $y$, $y'$, $y''$, or $y'''$ is raised to a power other than 0 or 1, and the coefficients of each term containing $y$ or one of its derivatives may all be viewed as functions of $x$.

In the first example,

$$A_3(x) = 2, A_2(x) = 0, A_1(x) = 3, A_0(x) = -\frac{1}{2} \text{ and } F(x) = x^3.$$

In the second example,

$$A_2(x) = x, A_1(x) = -3x^2, A_0(x) = -\frac{1}{x} \text{ and } F(x) = 0.$$

**Key Point.** In this chapter, we will only discuss solution methods for higher-order equations that are also linear.

To each linear differential equation, we will associate a homogeneous equation. For our purposes, homogeneous simply means that there are no non-zero terms in the equation that do not involve $y$ or one of its derivatives: in general, the homogeneous equation associated to

$$y^{(n)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

is

$$y^{(n)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

which we obtain by simply erasing the $f(x)$ term.
Example. The homogeneous equation associated to
\[ 2y''' + 3y' - \frac{y}{2} = x^3 \]
is
\[ 2y''' + 3y' - \frac{y}{2} = 0. \]

We will spend a great deal of time studying linear equation that have constant coefficients, such as
\[ 2y''' + 3y' - \frac{y}{2} = x^3; \]
the coefficients of \( y \) and of each its derivatives are constants, as opposed to the equation
\[ xy'' - 3y = \sin x, \]
which has nonconstant coefficients since the coefficient of \( y'' \) is the variable \( x \). A linear \( n \)th order differential equation with constant coefficients has form
\[ a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y = f(x), \]
where each \( a_i \) is a constant.

We have just seen many terms describing specific types of equations; these can get a bit confusing, so it would be appropriate to look at a variety of examples together. Let’s classify the following (second-order) equations using the definitions above:

\[ \begin{align*}
    y'' \sin y &= x & \quad xy'' + 3x^2y' - \frac{y}{x} &= 0 & \quad 2y'' + 3y' - \frac{y}{2} &= 0 \\
    (y'')^2 + y &= 0 & \quad xy'' + 3x^2y' - \frac{y}{x} &= x \cos x & \quad 2y'' + 3y' - \frac{y}{2} &= x^3
\end{align*} \]
Linear equations are simpler than non-linear ones; linear equations with constant coefficients are simpler than linear equations with nonconstant coefficients; and homogeneous equations are simpler than non-homogeneous ones. In the next section, we will deal specifically with the task of solving homogeneous linear equations with constant coefficients. Surprisingly, finding solutions for such equations can help us significantly with solving nonhomogeneous linear equations with constant coefficients.
In this chapter, we will focus on solving linear equations with constant coefficients; and as indicated in the previous paragraph, we will begin by studying linear homogeneous equations with constant coefficients (LHCC equations).

Typical Application of Higher-order Linear Homogeneous Equations with Constant Coefficients

Focusing our attention on linear homogeneous equations with constant coefficients may seem to be rather limiting, but it turns out that such equations show up quite often in modeling problems. One application in which such equations frequently occur is in the description of spring and shock-absorber systems. In such a system, an object with mass \( m \) is attached to a spring; the spring asserts a force \( F_S \) on the object, while a shock-absorber on the opposite side provides a resisting force \( F_R \):

Notice that we can describe the position of the object as a function of time, say \( x(t) \), so that the object's velocity is \( x'(t) \), and its acceleration is \( x''(t) \). The object has an equilibrium position, which we think of as \( x = 0 \). We would like to determine the unknown position function \( x \), which we can do using Newton's second law

\[ F = ma. \]

Recall that \( F \) is the total force acting on the object, \( m \) is the object's mass, and \( a = x''(t) \) its acceleration. Assuming that \( F_S \) and \( F_R \) are the only two forces acting on the object, we write

\[ F_R + F_S = mx''. \]

Of course, we need to describe \( F_S \) and \( F_R \) explicitly; we can assume that the force \( F_S \) that the spring asserts on the object is proportional to the displacement (i.e. position \( x(t) \)). When the object is on the right side of the equilibrium position, \( F_S \) acts to pull it back to equilibrium; so

when \( x > 0 \), \( F_S < 0 \).

On the other hand, when the object is on the left side of its equilibrium position, \( F_S \) acts to push it back to equilibrium; so

when \( x < 0 \), \( F_S > 0 \).

Putting this information together, we write

\[ F_S = -kx, \text{ where } k > 0. \]
As for $F_R$, we assume that the force supplied by the shock-absorber is proportional to the velocity $x'(t)$ of the object. In addition, the shock-absorber always acts in the direction opposite of the motion of the object, so we may write

$$F_R = -cx'(t), \text{ where } c > 0.$$ 

Now that we have described $F_R$ and $F_S$, we arrive at a differential equation relating $x$, $x'$, $x''$, and $t$:

$$-cx' - kx = mx'', \text{ or }$$

$$mx'' + cx' + kx = 0.$$ 

Since each of $m$, $c$, and $k$ are constants, this is a linear homogeneous differential equation with constant coefficients.

If $F_S$ and $F_R$ are not the only forces acting on the object—i.e., there is some other force $F(t)$ involved—then we must account for this force in Newton’s law, writing

$$F_R + F_S + F(t) = mx''.$$ 

With the same assumptions on $F_R$ and $F_S$, this means that

$$mx'' + cx' + kx = F(t).$$ 

Notice that this equation is a linear non-homogeneous equation with constant coefficients—in particular, the equation

$$mx'' + cx' + kx = 0$$

above is the homogeneous equation associated to

$$mx'' + cx' + kx = F(t).$$ 

This example provides clear motivation as to why we might wish to learn to solve both linear nonhomogeneous equations with constant coefficients, as well as the associated homogeneous equations.

In reference to point (2) on page 3 of this document, we have seen that the general solution to a first-order differential equation is a function that includes an arbitrary constant $C$. As one might expect, solutions of higher-order equations are more complicated. Let’s look at an example to help us see what issues come up.

**Example.** The equation $y'' + y = 0$ is a second-order LHCC equation.

1. Show that $v_1 = \sin x$ and $v_2 = \cos x$ are solutions to the equation above.
2. Show that $V = Av_1 + Bv_2$, where $A$ and $B$ are constants, is a general solution to the equation.
3. Find the particular solution so that $V(\pi/2) = 3$ and $V'(\pi/3) = 1/2$. 

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1. If $v_1 = \sin x$, then $v'_1 = \cos x$ and $v''_1 = -\sin x$; so clearly
\[ v'' + v = \sin x - \sin x = 0, \]
and $v_1$ is a solution. Similarly, $v_2 = \cos x$, $v'_2 = -\sin x$ and $v''_2 = -\cos x$, and
\[ v''_2 + v_2 = -\cos x + \cos x = 0 \]
so that $v_2$ is also a solution.

2. If
\[ V = Av_1 + Bv_2 = A\sin x + B\cos x, \]
then
\[ V' = A\cos x - B\sin x, \]
and
\[ V'' = -A\sin x - B\cos x. \]

Clearly
\[ V'' + V = -A\sin x - B\cos x + A\sin x + B\cos x = 0, \]
so
\[ V = A\sin x + B\cos x \]
is indeed a general solution to the equation.

3. Notice that, since there are \textit{two} arbitrary constants in our solution, we must specify \textit{two} pieces of data about our particular solution function if we hope to find $A$ and $B$–one is not enough!

We can find $A$ using the first condition: since $V(\pi/2) = 3$, we know that
\[ 3 = A\sin(\pi/2) + B\cos(\pi/2) = A + 0 \]
so that $A = 3$.

Now we may use the second condition to find $B$: since $V'(\pi/3) = 1/2$, we know that
\[ \frac{1}{2} = 3\cos(\pi/3) - B\sin(\pi/3) = 3 - \frac{B\sqrt{3}}{2}, \]
so that $B = 2/\sqrt{3}$. Thus the particular solution to the initial value problem
\[ y'' + y = 0, \quad y(\pi/2) = 3, \quad y'(\pi/3) = 1/2 \]
is
\[ V = 3\sin x + \frac{2\cos x}{\sqrt{3}}. \]
We know that (assuming that it is well behaved) a first-order equation has a single solution; and we have now seen a second-order equation for which we were able to find two essentially different solutions whose sum was another solution. Following this pattern, you may have guessed the following fact:

**Key Point.** The $n$th order linear homogeneous equation with constant coefficients

\[ a_n y^{(n)} + a_{n-1} y^{n-1} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0 \]

has $n$ essentially different solutions $y_1, y_2, \ldots, y_n$, and

\[ Y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n, \]

where each $c_i$ is an arbitrary constant, is the most general solution to the equation.

We will discuss the phrase essentially different in a later section, but the following example provides motivation for the phrases’ meaning:

The functions $w_1 = 2 \sin x$ and $w_2 = -3 \sin x$ are solutions to the differential equation $y'' + y = 0$, but they are not “essentially different” in the way that $v_1 = \sin x$ and $v_2 = \cos x$ are. In particular, $w_1$ is a constant multiple of $w_2$,

\[ w_1 = -\frac{2}{3} w_2, \]

but $v_1 = \sin x$ is certainly not a constant multiple of $v_2 = \cos x$.

Given that a general solution to an $n$th order LHCC equation will have $n$ arbitrary constants, you may have already guessed the following fact, which corresponds to point (3) on page 3 of this document:

**Key Point.** We must have $n$ initial conditions in order to find a particular solution to an $n$th order LHCC equation.

In other words, we must specify $n$ pieces of data in order to find the $n$ arbitrary constants that show up in the general solution; in the previous example, we were given two initial conditions to help us find the particular solution to the 2nd order equation.