Technique of Separation of Variables

In this section we will learn a technique for solving differential equations of a very special form. In particular, our first example will be to solve the equation

$$y' = \frac{x^2}{y + x^3 y}.$$

Notice that the entire right-hand side of the above equation can be factored as a *product* of a function involving only x, and a function involving only y:

$$\frac{x^2}{y + x^3 y} = \left(\frac{x^2}{1 + x^3}\right)\left(\frac{1}{y}\right).$$

This is such a helpful property that we will give it a name:

Definition. A differential equation is *separable* if it can be written in the form

$$y' = f(x)g(y).$$

The equation above is indeed separable, since we can think of

$$f(x) = \frac{x^2}{1 + x^3} \text{ and } g(y) = \frac{1}{y},$$

so that the original

$$y' = \frac{x^2}{y + x^3 y}$$

can indeed be written as

$$y' = f(x)g(y).$$

The equation

$$y' = x - y,$$

on the other hand, is *not* separable. There is no way to factor $x - y$ so that it looks like a product $f(x)g(y)$ of a function of x and a function of y.

The method of separation of variables is quite straightforward, but it can *only* be applied to separable equations. The separable equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

can be separated even more, by rewriting it (informally) as

$$\frac{\mathrm{d}y}{g(y)} = f(x)\ \mathrm{d}x$$

(so that only y's appear on the left, and only x's on the right). This equation is equivalent to

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\ \mathrm{d}x,$$

where the integral on the left-hand side is with respect to y, and the integral on the right-hand side is with respect to x (the equations are equivalent because two functions that have the same derivative on an interval only differ by a constant). If we can evaluate each integral, we can find a solution function y.

Example. Let's use the method to solve the initial value problem

$$y' = \frac{x^2}{y + x^3 y}, \ y(0) = -1.$$

We have already seen that the equation is separable, and can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x^2}{1+x^3}\right)\left(\frac{1}{y}\right).$$

Now we should rewrite the equation so that only y's appear on the left, and only x's on the right:

$$y \ \mathrm{d}y = \frac{x^2}{1+x^3} \ \mathrm{d}x,$$

which is equivalent to

$$\int y \ \mathrm{d}y = \int \frac{x^2}{1+x^3} \ \mathrm{d}x.$$

Finally, integrate each side with respect to its variable. The left-hand side is easy:

$$\int y \ \mathrm{d}y = \frac{y^2}{2} + c.$$

On the right-hand side, we'll need to use the u-substitution $u = 1 + x^3$, so that $\mathrm{d}u = 3x^2 \ \mathrm{d}x$:

$$\begin{aligned} \int \frac{x^2}{1+x^3} \ \mathrm{d}x &= \int \frac{1}{3u} \ \mathrm{d}u \\ &= \frac{1}{3} \ln|u| + C \\ &= \frac{1}{3} \ln|1+x^3| + C. \end{aligned}$$

So since

$$\int y \ \mathrm{d}y = \int \frac{x^2}{1+x^3} \ \mathrm{d}x,$$

we have

$$\frac{y^2}{2} = \frac{1}{3} \ln|1+x^3| + C$$

(C stands for any constant, so we can combine the constants on either side into just one on the right-hand side).

We can simplify a bit by writing

$$y = \pm\sqrt{\frac{2}{3} \ln|1+x^3| + C}.$$

Since this is an initial value problem, we can determine the value for C. You may be a bit worried about our solution at this point–our value above for the function y seems to indicate that we could have *two different solution functions.* However, recalling the theorem from the last section, we know that we should get a unique solution to this initial value problem since

$$\frac{x^2}{y(1+x^3)} \text{ and } \frac{-2x^2}{y^2(1+x^3)}$$

are both continuous near the initial value $(0,-1)$.

So let's find the value for C and see what happens. We know that $y(0)=-1$, so

$$\begin{aligned} -1 &= \pm\sqrt{\frac{2}{3}\ln|1|+C} \\ &= \pm\sqrt{0+C} \\ &= \pm\sqrt{C}. \end{aligned}$$

Now $\sqrt{C}=-1$ doesn't make any sense, so it is not a solution near the point $(0,-1)$. We must have $-\sqrt{C}=-1$ so that $C=1$. Thus the solution function that makes sense near $(0,-1)$ is

$$y=-\sqrt{\frac{2}{3}\ln|1+x^3|+1}.$$

Notice that, for most initial values, we *will* get a unique solution, either

$$y=\sqrt{\frac{2}{3}\ln|1+x^3|+C}$$

or

$$y=-\sqrt{\frac{2}{3}\ln|1+x^3|+C},$$

based on the choice of the initial value.

Example. Solve the equation

$$y'=\frac{1}{x^2y+xy+x^2+x}.$$

At first glance, this equation does not appear to be separable:

$$\frac{1}{x^2y+xy+x^2+x}$$

definitely doesn't look like a product of a function of x and a function of y. However, we can do some factoring, and since we have no other options, we might as well give it a try:

$$\begin{aligned}\frac{1}{x^2y+xy+x^2+x} &= \frac{1}{x(xy+y+x+1)}\\ &= \frac{1}{x(y(x+1)+x+1)}\\ &= \frac{1}{x(y(x+1)+(x+1))}\\ &= \frac{1}{x(x+1)(y+1)}\\ &= \frac{1}{y+1}\cdot\frac{1}{x(x+1)},\end{aligned}$$

so the equation is actually separable.

Thus we can solve using the techniques from this section: rewrite

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y+1}\cdot\frac{1}{x(x+1)} \text{ as}$$

$$(y+1)\ \mathrm{d}y = \frac{1}{x(x+1)}\ \mathrm{d}x.$$

Now we can solve by integration:

$$\int (y+1)\ \mathrm{d}y = \int \frac{1}{x(x+1)}\ \mathrm{d}x.$$

On the left-hand side, we have

$$\int (y+1)\ \mathrm{d}y = \frac{y^2}{2} + y + c.$$

To solve the integral

$$\int \frac{1}{x(x+1)}\ \mathrm{d}x,$$

we'll need to use partial fractions: rewrite the fraction

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

Adding the two fractions on the right, we see that

$$\frac{1}{x(x+1)} = \frac{A(x+1)+Bx}{x(x+1)}.$$

This means that

$$A(x+1)+Bx = 1,$$

and since there are no x's on the right, we have the system of equations

$$A+B=0,\quad A=1,$$

which is easy to solve: since $A = 1$, we know that $B = -1$. Thus

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$$

so that

$$\begin{aligned} \int \frac{1}{x(x+1)}\, \mathrm{d}x &= \int \frac{1}{x} - \frac{1}{x+1}\, \mathrm{d}x \\ &= \ln|x| - \ln|x+1| + C. \end{aligned}$$

Again, since

$$\int (y+1)\, \mathrm{d}y = \int \frac{1}{x(x+1)}\, \mathrm{d}x,$$

our solution is given by

$$\frac{y^2}{2} + y = \ln|x| - \ln|x+1| + C.$$

Notice that the solution above cannot be rewritten so that y is in terms of x only, say as $y = k(x)$. We can try factoring a copy of y from the left-hand side, but there will still be a factor of $(y/2 + 1)$ left over.

It is better to simply leave the solution written in the form

$$\frac{y^2}{2} + y - \ln|x| + \ln|x+1| = C.$$

This solution may not be completely satisfying, since we generally prefer to have an explicit solution for the function y in terms of x; however, it is a perfectly acceptable answer, since the equation establishes a relationship between x and y. We will call such a solution an *implicit solution* to the equation.

Definition. A solution to a differential equation that is given in terms of an equation involving x and y, without defining y explicitly as a function of x, is called an *implicit solution.*

Example (Application)**.** A tank initially contains 100 gallons of a salt-water solution, with a total of 5 pounds of salt dissolved in the water. At time $t = 0$, a salt water solution containing $1/4$ pound of salt per gallon begins to flow into the tank at a rate of 2 gallons per minute, and the well-stirred mixture is allowed to drain from the tank at the same rate, so that the tank maintains 100 gallons of solution. How many pounds of salt will be in the tank after 25 minutes of this process?

Ideally, we would like to come up with a function $A(t)$, whose value at time t is the number of pounds of salt in the tank. We have some information about how the amount of salt in the tank changes over time–the change in salt over time (A') is only affected by the amount of salt added (A_i) and the amount of salt drained (A_o). In particular, the change in the amount of salt is the difference of these two quantities,

$$A' = A_i - A_o.$$

We would like to have specific functions for each of A_i and A_o.

The amount of salt being added to the tank is constant; 2 gallons of solution per minute is entering the tank, 1/4 pound of which is salt. So the amount of salt entering the tank at time t is

$$A_i = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

On the other hand, the amount of salt being drained will change over time. At time t, the portion of solution in the tank that is salt is

$$\frac{\text{current amount of salt in the tank}}{\text{number of gallons of solution in the tank}} = \frac{A}{100}.$$

Since the solution is draining at 2 gallons per minute, we have

$$A_o = 2 \cdot \frac{A}{100} = \frac{A}{50}.$$

Thus our differential equation is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} - \frac{A}{50} = \frac{1}{2}\left(\frac{25 - A}{25}\right),$$

which is clearly separable. Again, we can now solve the equation using the technique described above. Separating the variables, we have

$$\frac{25}{25 - A}\ \mathrm{d}A = \frac{1}{2}\ \mathrm{d}t,$$

which is equivalent to

$$\int \frac{25}{25 - A}\ \mathrm{d}A = \int \frac{1}{2}\ \mathrm{d}t.$$

Using the u-substitution $u = 25 - A,\ \mathrm{d}u = -\ \mathrm{d}A$, we have

$$\begin{aligned} \int \frac{25}{25 - A}\ \mathrm{d}A &= -\int \frac{25}{u}\ \mathrm{d}u \\ &= -25 \ln |u| + c \\ &= -25 \ln |25 - A| + c_1. \end{aligned}$$

On the right-hand side,

$$\int \frac{1}{2}\ \mathrm{d}t = \frac{t}{2} + c_2.$$

So

$$-25 \ln |25 - A| = \frac{t}{2} + c_2,$$

and with some simplification we have

$$A = 25 - Ce^{-t/50}.$$

We can find C since we know that the amount of solution in the tank at $t = 0$ is 5 pounds, i.e. $A(0) = 5$:

$$5 = 25 - C \text{ so that } C = 20.$$

Thus the solution function is

$$A = 25 - 20e^{-t/50},$$

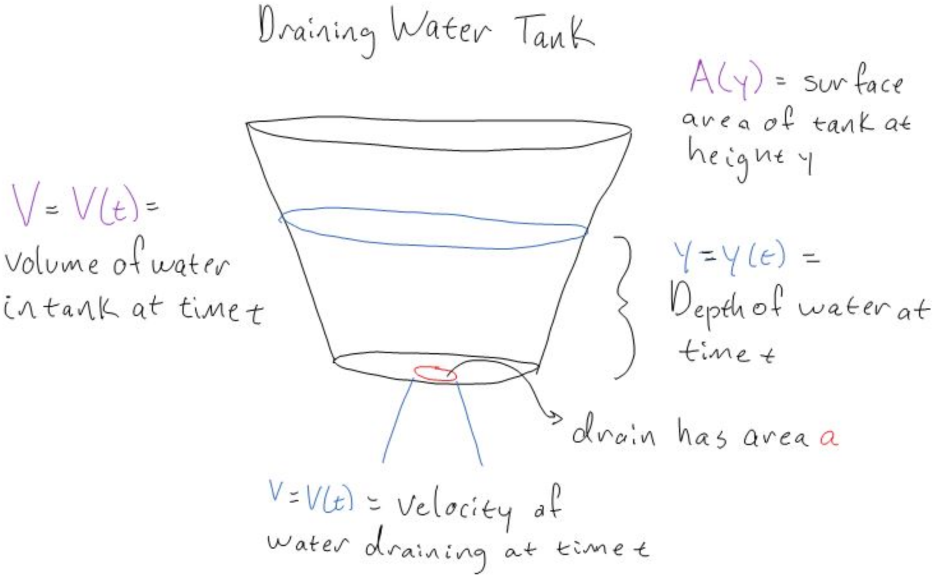
and we can solve the original problem: after 25 minutes of the process there will be

$$\begin{aligned} A(25) &= 25 - 20e^{-25/50} \\ &= 25 - 20e^{-1/2} \\ &\approx 12.9 \end{aligned}$$

pounds of salt in the tank.

Torricelli's Law

One application that comes up quite often is the draining tank problem. A tank, such as the one below, is filled with a liquid. At time $t = 0$, a drain in the bottom of the tank is opened, and the tank begins to empty.



Variables involved in draining tank

There are many variables involved in a problem of this type:

$y = y(t)$	depth of liquid in tank at time t
a	surface area of drain (this is a constant)
v	velocity of liquid leaving tank at time t
$V = V(t)$	volume of liquid in tank at time t
$A = A(y)$	surface area of tank at depth y

Given a particular tank system, we would like to describe the system explicitly and answer related questions. For example, we will look at the following problem after we have looked at some background information about these types of systems:

The town of Milford, New Jersey, has a spherical water tank with radius 20 feet. The town manager just discovered that a prankster managed to dump a large amount of green dye in the tank, and now the tank must be emptied and scrubbed. The (circular) drain in the bottom of the tank has a radius of 1 foot. Given that the tank was full when the manager discovered the problem, how long will it take to drain the tank?

Let's try to understand the general setup of these problems before tackling a specific example. We would like to find ways to relate the variables above. There is a well-known relationship between the velocity v of draining water and the depth y of liquid in the tank:

$$v = \sqrt{2gy},$$

where g is acceleration due to gravity ($9.8\ m/s^2$ or $32.2\ ft/s^2$).

Notice that the law allows us to think of v as a function of the depth y of the liquid, as opposed to thinking of it as a function of time t (since v and y are both actually function of t; in a sense we're just hiding the ts until we need them). We will use this trick several times.

Another important piece of information that we will use is Torricelli's Law, which relates the change in volume of the liquid, $\frac{\mathrm{d}V}{\mathrm{d}t}$, to the depth y of the liquid. Notice that the volume of the liquid draining from the tank at time t (i.e., when the liquid's height is $y = y(t)$) is

$$a\sqrt{2gy},$$

the product of the area a of the drain with the velocity $v = \sqrt{2gy}$ of the draining liquid. Torricelli's Law says that this quantity is a good description of $\frac{\mathrm{d}V}{\mathrm{d}t}$:

$$\mathbf{Torricelli'sLaw}: \qquad \frac{\mathrm{d}V}{\mathrm{d}t} = -a\sqrt{2gy}.$$

Now we have a relationship between y and $\frac{\mathrm{d}V}{\mathrm{d}t}$. But since y is also a function of t, we can actually get more information. Let's think of the rate of change of V with respect to depth y, as opposed to time t–in other words, consider

$$\frac{\mathrm{d}V}{\mathrm{d}y} \text{ instead of } \frac{\mathrm{d}V}{\mathrm{d}t}.$$

From Calculus 2, we know that we can calculate the volume of the tank as the integral of its area $A(y)$:

$$V(y) = \int_0^y A(\gamma)\ \mathrm{d}\gamma,$$

so that

$$\frac{\mathrm{d}V}{\mathrm{d}y} = A(y)$$

(from the Fundamental Theorem of Calculus).

Since y is a function of t, we can expand $\frac{\mathrm{d}V}{\mathrm{d}t}$ as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = A(y)\frac{\mathrm{d}y}{\mathrm{d}t} = A(y)y'.$$

Let's put all of our information together: we know that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -a\sqrt{2gy},$$

and also that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = A(y)\frac{\mathrm{d}y}{\mathrm{d}t} = A(y)y'.$$

From this we get the extremely useful equation

$$A(y)y' = -a\sqrt{2gy},$$

where

$y = y(t)$	depth of liquid in tank at time t
a	surface area of drain
$A = A(y)$	surface area of tank at depth y.

Example (Application)**.** The town of Milford, New Jersey, has a spherical water tank with radius 20 feet. The town manager just discovered that a prankster managed to dump a large amount of green dye in the tank, and now the tank must be emptied and scrubbed. The (circular) drain in the bottom of the tank has a radius of 1 foot. Given that the tank was full when the manager discovered the problem, how long will it take to drain the tank?

We will use the equation derived above,

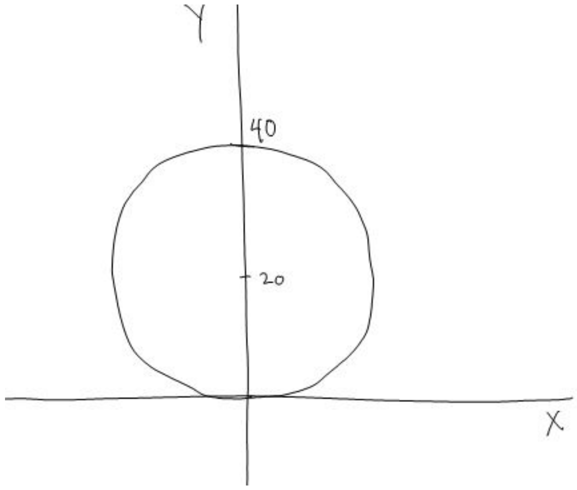
$$A(y)y' = -a\sqrt{2gy},$$

to solve the problem. In order to use the equation, we'll need to specify the unknowns $A(y)$ (surface area of tank at depth y) and a (surface area of the drain).

It's easy enough to find a: since the radius of the drain is 1 foot, its surface area is

$$a = \pi r^2 = \pi.$$

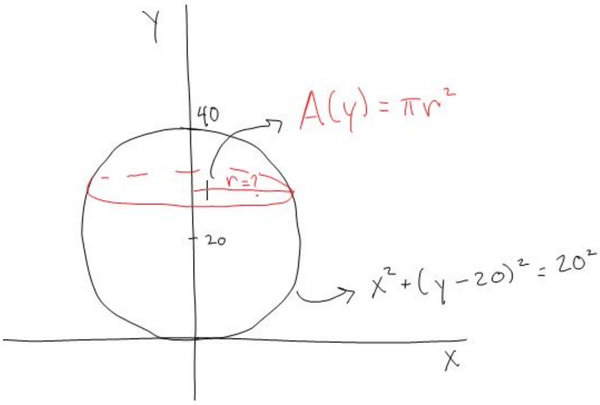
It will take a bit more effort to determine a formula for $A(y)$. Let's start with a picture, placing the bottom of the tank at the origin on the xy plane:



We can easily write an equation for the circle we see above: it is described by

$$x^2 + (y - 20)^2 = 20^2.$$

We need a formula for $A(y)$. Since $A(y)$ is just a cross section of a sphere (i.e., a circle!), all we need to do is find the radius of a typical slice:



Notice that the radius of the slice above is just the x component of a point on the circle $x^2 + (y - 20)^2 = 20^2$. So we can get r by solving for x: at a particular choice of y,

$$x = \sqrt{20^2 - (y - 20)^2} = \sqrt{40y - y^2}.$$

So the radius of the slice is

$$r = \sqrt{40y - y^2},$$

and we can now calculate $A(y)$:

$$A(y) = \pi r^2 = \pi(40y - y^2).$$

Now we can use the original formula $A(y)y' = -a\sqrt{2gy}$: we know that

$$\pi(40y - y^2)y' = -\pi\sqrt{2gy},$$

which simplifies to

$$(40y - y^2)\frac{\mathrm{d}y}{\mathrm{d}t} = -\sqrt{2gy}.$$

This is clearly separable, so we will proceed as with the other examples in this section: rewriting the equation as

$$(40y^{1/2} - y^{3/2})\ \mathrm{d}y = -\sqrt{2g}\ \mathrm{d}t,$$

we may now integrate:

$$\int (40y^{1/2} - y^{3/2})\ \mathrm{d}y = \int -\sqrt{2g}\ \mathrm{d}t.$$

The right-hand side is simple:

$$\int -\sqrt{2g}\ \mathrm{d}t = -\sqrt{2g}t + C.$$

The left-hand side works out nicely as well:

$$\int (40y^{1/2} - y^{3/2}) \, \mathrm{d}y = \frac{80}{3}y^{3/2} - \frac{2}{5}y^{5/2}.$$

Our solution function is given implicitly by the equation

$$\frac{80}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\sqrt{2g}t + C.$$

We can determine the value for C using the given information: at time $t = 0$, the tank was full. This translates into mathematics as

$$y(0) = 40,$$

so we have

$$\frac{80}{3}(40)^{3/2} - \frac{2}{5}(40)^{5/2} = 0 + C.$$

Solving for C gives us $C = 2698.48$, so our solution equation is

$$\frac{80}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\sqrt{2g}t + 2698.48$$

We were asked to determine how long it would take to drain the tank, which means that we wish to find the time t so that $y(t) = 0$. This gives us the equation

$$0 = -\sqrt{2g}t + 2698.48.$$

Using $g = 32.2$ and solving for t, we get $t \approx 336.3$ seconds, or 5.6 minutes.