**The Fundamental Theorem for Line Integrals**

For the purposes of this section, we will start by recording a few definitions.

A curve is *simple* if it does not self-intersect. The first example below is a simple curve, while the second intersects itself, thus is not simple:

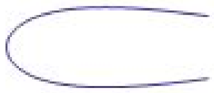


Figure 1: Simple Curve

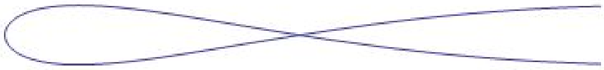


Figure 2: Non-simple Curve

A curve is *closed* if its initial and terminal points are the same. The first example below is a closed curve, while the second is not:

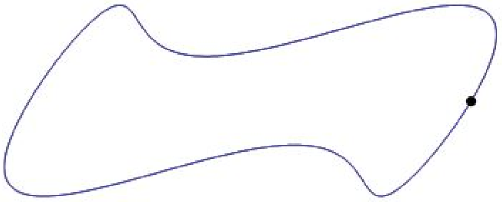


Figure 3: Closed Curve

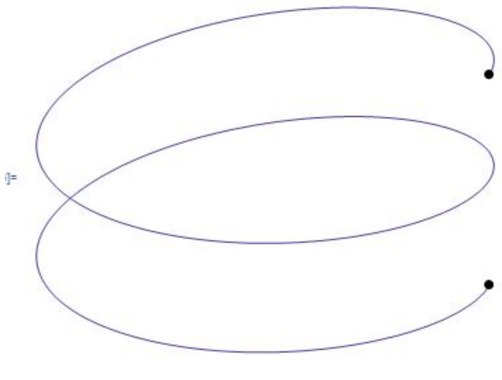


Figure 4: Non-closed Curve

We can combine the two definitions, as indicated in the following example:

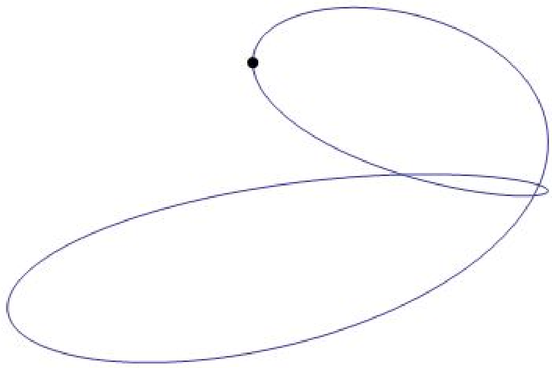
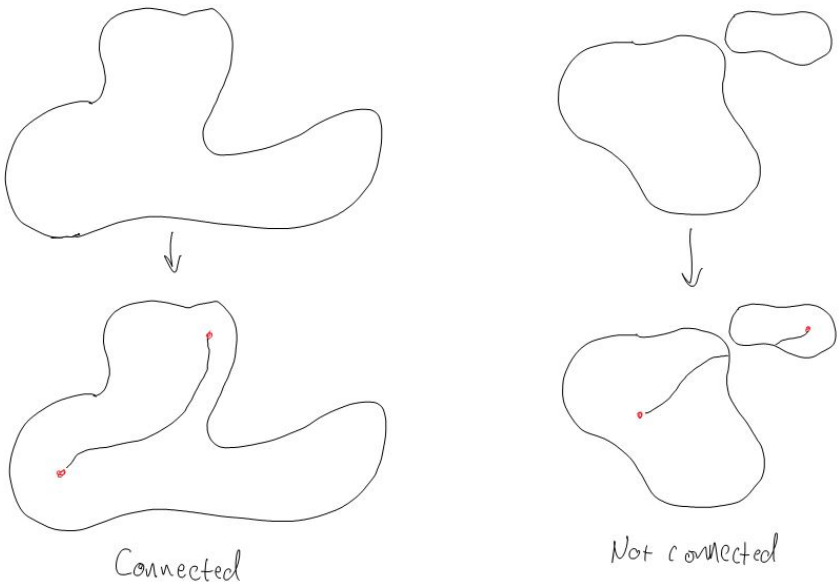


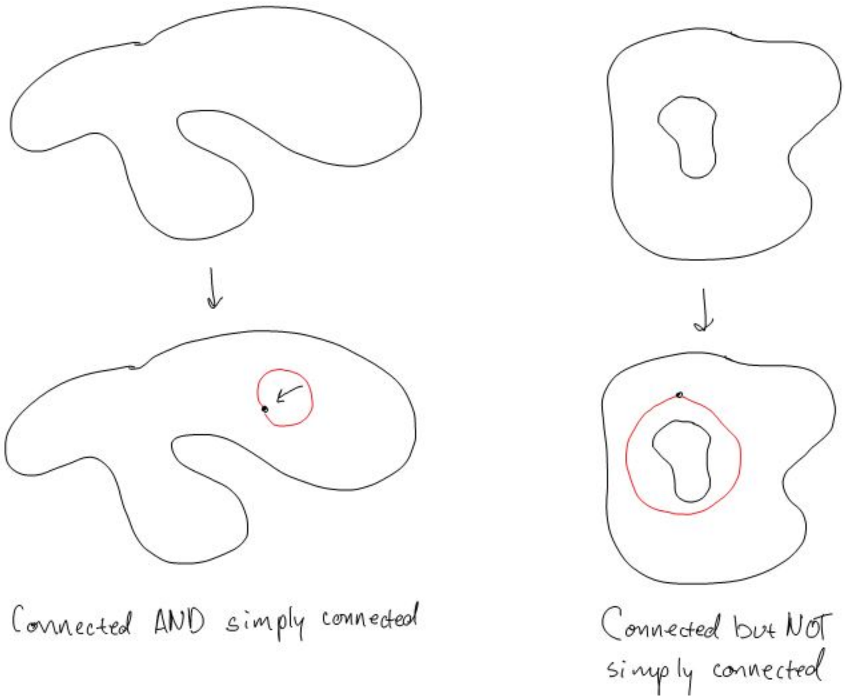
Figure 5: Non-simple, Closed Curve

Figure 4 above is an example of a non-simple, non-closed curve; figure 3 is a simple, closed curve; and figure 1 is a simple, non-closed curve.

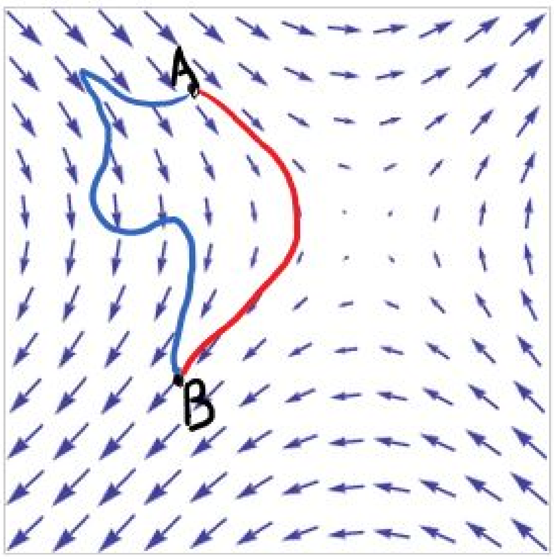
A region $R$ in space is *connected* if any two points in the region can be connected by a curve that lies completely within $R$. For example, the first region below is connected, but the second region is not:




A connected region $R$ is *simply connected* if (roughly speaking) $R$ has no holes; more precisely, the region is simply connected if any loop drawn in $R$ can be shrunk to a point without leaving $R$. The first region below is simply connected, while the second region is not:




In the previous section, we calculated the work done by a force in moving a particle along a path $C = \vec{r}(t)$ from $t = a$ to $t = b$. One question that arises is this: how is work affected if we choose to calculate it along a *different path* from $t = a$ to $t = b$? For example, consider calculating the work done in moving a particle from $t = a$ to $t = b$ along the two different curves in the vector field below:




It appears that the work done along the red curve will probably be less than the work along the blue curve. In general, it seems that there is probably no connection between the work done along two curves that share starting and ending points.

However, for *select* vector fields, a remarkable property holds: given points $A$ and $B$ in the field, the work done in moving from $A$ to $B$ is constant, and does not depend on the choice of path from $A$ to $B$.

**Definition 0.0.1.** Suppose that $\vec{F}$ is a vector field on a region $D$ in space, and let $A$ and $B$ be points in $D$. If the work $W = \int_A^B \vec{F} \cdot d\vec{r}$ done by $\vec{F}$ in moving a particle from $A$ to $B$ is not dependent upon the path chosen between $A$ and $B$, we say that $\vec{F}$ is a *conservative vector field* on $D$, and that the integral $\int \vec{F} \cdot d\vec{r}$ is *path independent* in $D$.

The important question to ask here is this: Can we describe all of the vector fields that are conservative fields? The answer turns out to be quite nice (we will state and discuss this in more detail in Theorems 2 and 4): a vector field $\vec{F}$ is conservative *if and only if* it is the gradient field for some scalar function $f$, i.e. $\nabla f = \vec{F}$. Accordingly, we have the following definition:

**Definition 0.0.2.** Suppose that $\vec{F}$ is a vector field on the region $D$ in space. If there is a scalar function $f$ defined on $D$ so that

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \vec{F},$$

we say that $f$ is a *potential function* for $\vec{F}$.

In a sense, $\vec{F}$ plays the role of the derivative of $f$. Likewise, there is a sense in which we can think of $f$ as an antiderivative of $\vec{F}$. These ideas are not precise, since derivatives of scalar functions are again scalar functions, but the idea should help you to remember (and correctly interpret) the theorems to come.

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The theorem below corresponds to the Fundamental Theorem of Calculus, which tells us how to evaluate definite integrals of functions.

**Theorem 2. The Fundamental Theorem for Line Integrals**

Let $r(\vec{t})$ be a vector function that traces out the smooth curve $C$ joining points the terminal points of $r(\vec{a})$ and $r(\vec{b})$ on $a \leq t \leq b$. Let $f$ be a differentiable function that is the potential function for the vector field $\vec{F}$, i.e. $\nabla f = \vec{F}$ on a domain containing $C$. Then

$$W = \int_C \vec{F} \cdot d\vec{r} = f(r(b)) - f(r(a)).$$

The theorem says that certain types of line integrals are quite similar to single-variable integrals; if $\vec{F}$ has a potential function $f$ (which we interpret as an antiderivative), then to evaluate the line integral of $\vec{F}$ along the curve traced out by $r(\vec{t})$ we merely need to plug in the bounds $r(\vec{a})$ and $r(\vec{b})$ into $f$; the integral is $f(r(\vec{b})) - f(r(\vec{a}))$, and there is no need to go through the tedium of actually integrating.

Notice that the theorem does not apply to *every* vector field. If $\vec{F}$ is a vector field that *does not* have a potential function, i.e. $\vec{F}$ is not a gradient field, then we cannot use the theorem, but must make the calculation directly. We will record one more helpful theorem:

**Theorem 4.** Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field whose components $M$, $N$, and $P$ are continuous throughout an open connected region $D$ in space. Then $\vec{F}$ is conservative if and only if $\vec{F}$ is a gradient field for some differentiable scalar function $f$, i.e. $\nabla f = \vec{F}$.

We have seen quite a few different statements that all amount to the same thing; for sake of clarity, we record them below. The following statements are equivalent:

- $\vec{F}$ is a conservative vector field
- The integral $W = \int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path joining $A$ and $B$
- $\vec{F}$ has a potential function $f$ (i.e. $\nabla f = \vec{F}$)
- $W = \displaystyle\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$
- If $C$ is a closed loop, then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0$.

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**Example**

Let $f(x, y, z) = \ln(xyz)$, and let $\vec{F} = \nabla f$. Determine the work done by $\vec{F}$ in moving a particle along a smooth curve joining the points $(\frac{1}{4}, 2, 2)$ and $(1, e, 1)$.

Since $\vec{F}$ is the gradient field for $f$, i.e. $\vec{F}$ has a potential function, we know that $\vec{F}$ is a conservative field. Thus by Theorem 2, we know that the work is

$$W = \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

So instead of finding $\vec{F}$ and evaluating an integral, we only need to calculate

$$f(1, e, 1) - f(\frac{1}{4}, 2, 2) = \ln(1 \cdot e \cdot 1) - \ln(\frac{1}{4} \cdot 2 \cdot 2) = \ln e - \ln 1 = 1.$$

Thus $W = 1$.

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In practice, there are several different ways to show that a field is conservative. We could find the potential function $f$ for $\vec{F}$, or we could show that the work integral for $\vec{F}$ is path independent. However, the quickest way to determine if $\vec{F}$ is conservative is to use the following test; the first is for vector fields in two dimensional space, the second for vector fields in three dimensional space:

**Component Test for Conservative Fields**

Let $\vec{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field on a region $D$, and assume that the component functions $P$ and $Q$ have continuous first partial derivatives. Then $\vec{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Let $\vec{F} = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$ be a vector field on a region $D$ that is simply connected, and assume that the component functions $M$, $N$, and $P$ have continuous first partial derivatives. Then $\vec{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

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**Examples**

Is the vector field $\vec{F} = 3x^2 \sin y\vec{i} - x^2 z\vec{j} - \sin y\vec{k}$ conservative?

We simply need to check the conditions from the test: we have $M(x, y, z) = 3x^2 \sin y$, $N(x, y, z) = -x^2 z$, and $P(x, y, z) = -\sin y$. Since $\dfrac{\partial P}{\partial y} = -\cos y$ and $\dfrac{\partial N}{\partial z} = -x^2$, we automatically know that the field is not conservative.

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Given $\vec{F}(x, y, z) = y^2\vec{i} + (2xy + e^{3z})\vec{j} + 3ye^{3z}\vec{k}$, show that the field is conservative and find a potential function for $\vec{F}$.

To show that the field is conservative, we need to check the partials. Since $M = y^2$, $N = 2xy + e^{3z}$, and $P = 3ye^{3z}$, we have

$$\frac{\partial P}{\partial y} = 3e^{3z} = \frac{\partial N}{\partial z},$$
$$\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \text{ and}$$
$$\frac{\partial N}{\partial x} = 2y = \frac{\partial M}{\partial y}.$$

Thus the field is indeed conservative.

Finding a potential function amounts to working backwards; we already know the partials of $f$ with respect to $x$, $y$, and $z$; we can integrate (along with a few other steps) to determine the actual value for $f$.

The potential function $f(x, y, z)$ is a scalar function so that

$$\nabla f = \vec{F}(x, y, z) = y^2\vec{i} + (2xy + e^{3z})\vec{j} + 3ye^{3z}\vec{k}.$$

In particular,

$$\frac{\partial f}{\partial x} = y^2, \ \frac{\partial f}{\partial y} = 2xy + e^{3z}, \text{ and } \frac{\partial f}{\partial z} = 3ye^{3z}.$$

To determine the actual value for $f(x, y, z)$, let's integrate each of the functions above. Keep in mind that, when we differentiate $f$ with respect to $x$, any term of $f$ that does not contain an $x$ will go to 0; so when we go "backwards" by integrating with respect to $x$, we will not be able to recover the function $g(y, z)$ that was killed by the differentiation; we will need to find a way to recover it. Since $\frac{\partial f}{\partial x} = y^2$, we know that the integral of $\frac{\partial f}{\partial x}$ with respect to $x$ is $xy^2 + g(y, z)$, i.e.

$$f(x, y, z) = xy^2 + g(y, z).$$

Differentiating $f$ with respect to $y$ gives us

$$\frac{\partial f}{\partial y} = 2xy + \frac{\partial}{\partial y} g(y, z);$$

note, however, that we determined above that

$$\frac{\partial f}{\partial y} = 2xy + e^{3z}.$$

Now the partial derivatives with respect to $y$ should match up, so

$$\frac{\partial f}{\partial y} = 2xy + \frac{\partial}{\partial y} g(y,z) = 2xy + e^{3z},$$

and we see that

$$\frac{\partial}{\partial y} g(y,z) = e^{3z}.$$

Integrating this last term with respect to $y$, we see that $g(y,z) = ye^{3z} + h(z)$. Thus

$$f(x,y,z) = xy^2 + g(y,z) = xy^2 + ye^{3z} + h(z).$$

Differentiating with respect to $z$, we see that

$$\frac{\partial}{\partial z} f(x,y,z) = 3ye^{3z} + h'(z);$$

but again, we already know that

$$\frac{\partial f}{\partial z} = 3ye^{3z},$$

so we see that

$$3ye^{3z} + h'(z) = 3ye^{3z}.$$

Thus $h'(z) = 0$, which means $h(z) = C$ for some constant $C$, and we have

$$f(x,y,z) = xy^2 + ye^{3z} + C.$$

This is the potential function for $\vec{F}$.