Triple Integrals

Just as we can evaluate the double integral of a function $f(x, y)$ of two variables over a region R in the xy plane, we can evaluate the *triple integral* of a function $f(x, y, z)$ of *three* variables over a region D in xyz space.

If $f(x, y)$ lies above the xy plane over the region R, we can think of $\int\int_R f(x, y)dA$ as the volume under f over the plane. Similarly, if $f(x, y, z)$ lies "above" the xyz hyperplane over the three dimensional region D, we can think of $\int\int\int_D f(x, y, z)dV$ as the hypervolume under $f(x, y, z)$ over the xyz hyperplane.

For the obvious reasons, this interpretation of a triple integral is quite difficult to work with; fortunately, we will usually only be concerned with abstract integrals, and not with actually interpreting the results. Indeed, we actually only need to be able to sketch the region D in three dimensional space in order to set up the bounds for the integral; the shape of the four dimensional graph of $f(x, y, z)$ will not present us with difficulties. To find the bounds for a triple integral, we will use a procedure (outlined below) quite similar to the procedure for setting up the bounds for a double integral. Before considering this problem, let's first look at a few theorems:

Theorem 7. Let E the region in three dimensional space that consists of the set of all points (x, y, z) so that $u_1(x, y) \leq z \leq u_2(x, y)$, $g_1(x) \leq y \leq g_2(x)$, and $a \leq x \leq b$. If $f(x, y, z)$ is continuous over the region E, then the triple integral of f over E is the iterated integral

$$\int\int\int_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdydx.$$

Although not explicitly stated, we may arrange the variables in the integral in any order that we choose, as long as the bounds are set up correctly. For example, if we wish to integrate first with respect to y, then the bounds on the inner integral should be functions of x and z, say $j_1(x, z)$ and $j_2(x, z)$.

We may evaluate triple integrals just as we did double integrals; first integrate with respect to the inner variable, treating all others as constants, then integrate with respect to the middle variable, then finally with respect to the outer variable.

In section 15.3, we saw that double integrals could be used to evaluate *areas* of regions in the xy plane. Specifically, to determine the area of the region D, we set up the integral $\int\int_D 1\ dA$.

Similarly, we can use triple integrals to think about *volumes* of regions in xyz space. If we want to know the volume of the region E, we simply integrate the function 1 over E. Specifically, we have the following theorem:

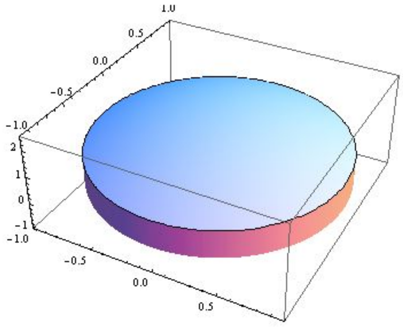
Theorem 0.0.1. The volume of a closed bounded region E in three dimensional space is

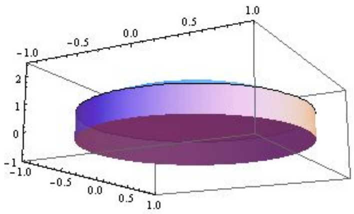
$$V = \int\int\int_E 1\ dV.$$

Determining the bounds of a triple integral

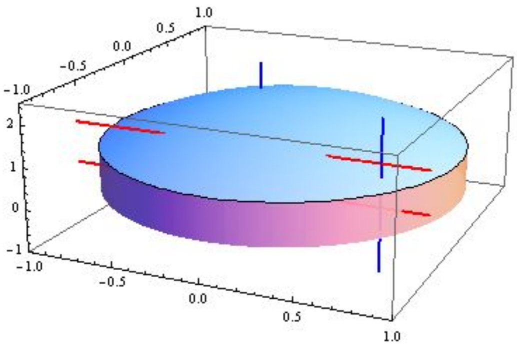
The procedure we will use to find the bounds of a triple integral is quite similar to the way that we determined bounds for double integrals. To begin with, we will need to make a choice: which variable should we integrate first?

To answer the question, we will need to sketch a graph of the region D. Once this is done, try considering one variable at a time; look for a variable so that the "lower bound" and "upper bound" with respect to that variable each consist of one function of the other two variables. The easiest way to do this is to draw a line through the region parallel to the axis whose variable you are studying; regardless of where you draw the line, it should start on the function g_1 and stop on the function g_2. For example, consider the region below, which is bounded above by $f(x, y) = 2 - x^2 - y^2$, below by $z = 0$, and on the sides by $x^2 + y^2 = 1$:





Notice that lines parallel to the z axis always enter through the $z = 0$ on the bottom and exit through the curved surface $f(x, y)$ at the top, while a line parallel to the x axis may exit through either the cylindrical surface, or through $f(x, y)$:

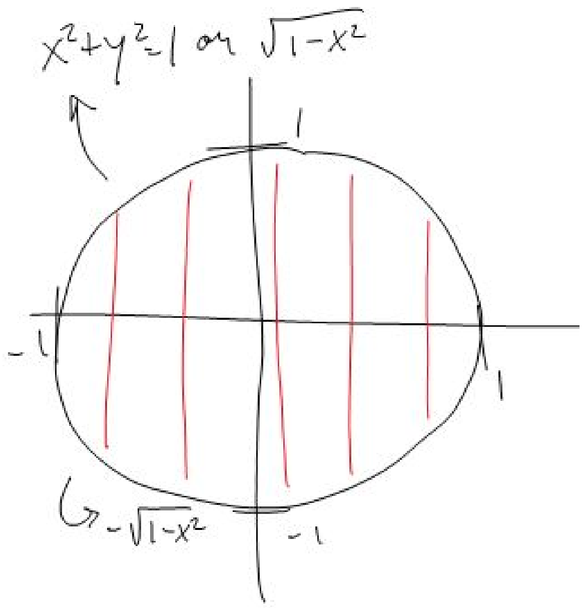


Since the bounds are consistent in the z direction, it will be easiest to integrate first with respect to z.

While this problem is easiest to set up by integrating first with respect to z, everything that follows is still applicable to problems where we integrate first with respect to x or y, with the obvious replacements. We find the bounds on z by finding the functions $g_1(x, y)$ and $g_2(x, y)$ that bound lines through the region parallel to the z axis; in this case, z goes from 0 to $2 - x^2 - y^2$.

Next we need to find the bounds on x and y; to do so, consider the "shadow" of the region on the xy plane. Once we have done this, the problem of finding the bounds on x and y is exactly the same as in the double integral case.

In our case, the "shadow" is the disk centered at the origin with radius 1:



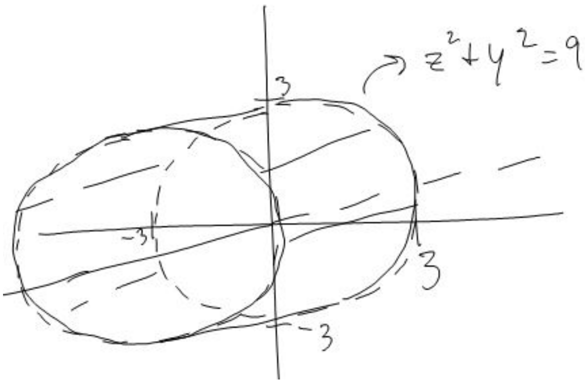
So the bounds on y are $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, and the bounds on x are -1 to 1. Thus the integral of the function $f(x, y, z)$ over the region D is given by

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{2-x^2-y^2} dzdydx.$$

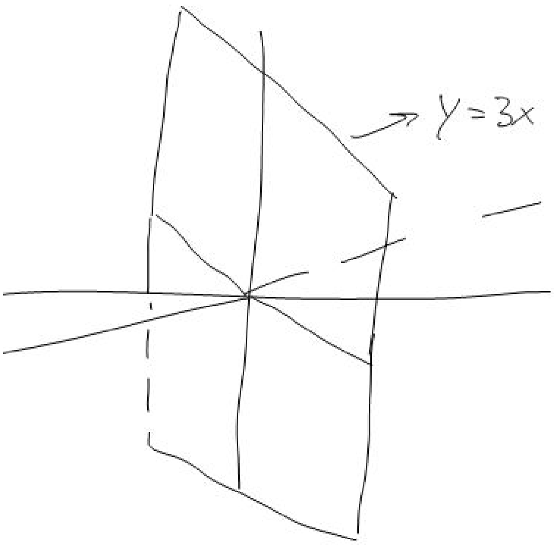
Examples

Evaluate $\int\int\int_D z\, dV$, where D is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant.

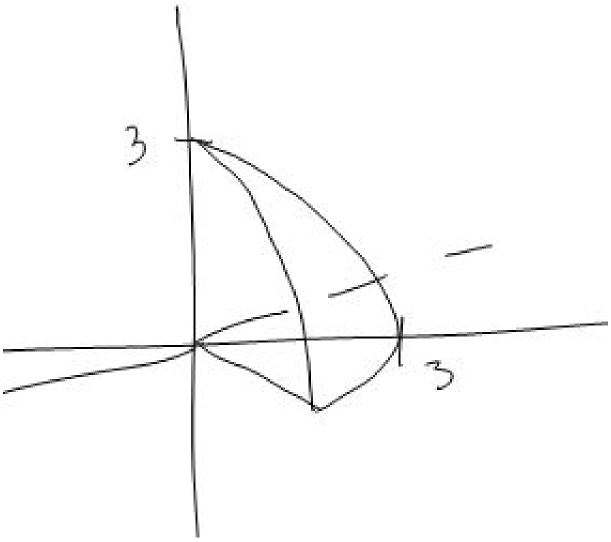
Let's graph $y^2 + z^2 = 9$. When $x = 0$ (i.e. on the yz plane), it is clear that $y^2 + z^2 = 9$ is just a circle of radius 3. However, the curve is unaffected by x, so for any choice of x, the picture will look the same; so we can see that $y^2 + z^2 = 9$ is the cylinder graphed below:



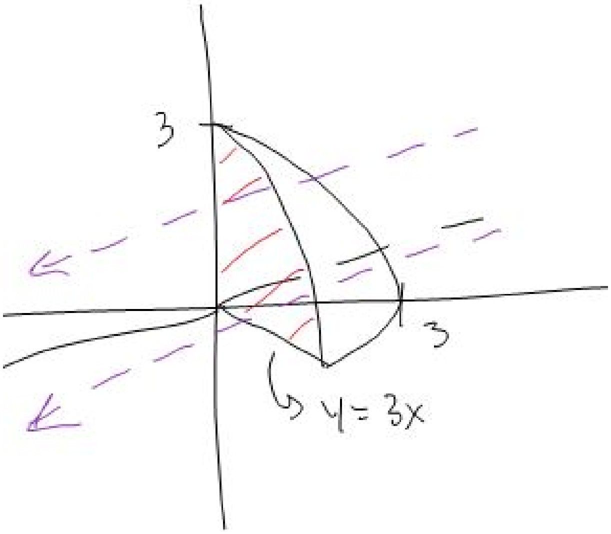
Similarly, to graph $y = 3x$ we note that when $z = 0$, $y = 3x$ is just a line in the xy plane; again, the choice of z does not affect the graph, so $y = 3x$ is the plane graphed below:



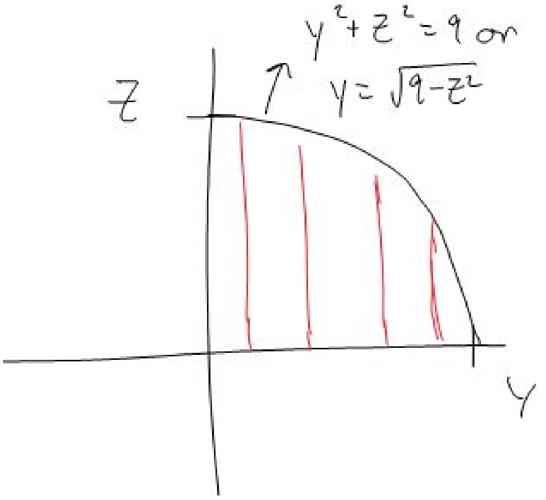
Since $z = 0$ is also a bound on the region, we do not have to worry about points below the xy plane. The plane $y = 3x$ intersects the surface to create a wedge:



Notice that lines through the region parallel to the x axis always enter the region through $x = 0$ and exit through $x = \frac{y}{3}$:



Now that we have bounds on x in terms of y and z, we are allowed to forget x and look at the "shadow" of the region in the yz plane. It is clear that we can integrate in either direction:



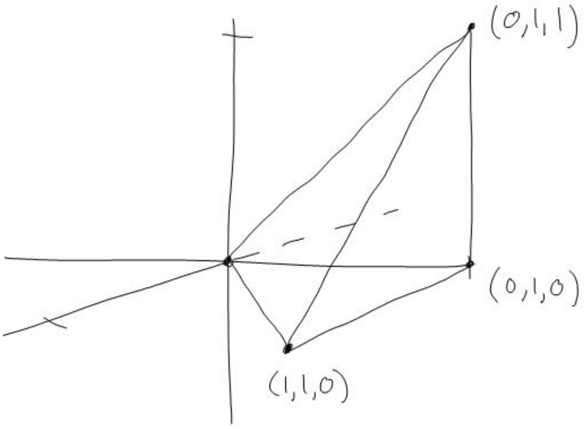
Let's integrate first with respect to y: the bounds will be from $y = 0$ to $y = \sqrt{9 - z^2}$:

The bounds on z are from 0 to 3, so we need to evaluate

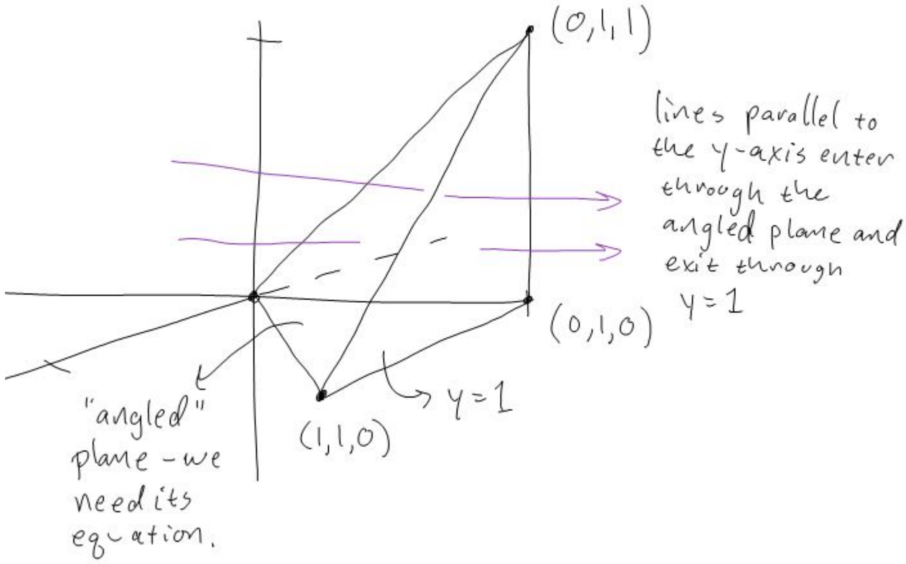
$$\begin{aligned}
\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\frac{y}{3}} z\ dxdydz &= \int_0^3 \int_0^{\sqrt{9-z^2}} (zx|_0^{\frac{y}{3}})\ dydz \\
&= \int_0^3 \int_0^{\sqrt{9-z^2}} \frac{yz}{3}\ dydz \\
&= \int_0^3 \frac{y^2 z}{6}|_0^{\sqrt{9-z^2}}\ dz \\
&= \int_0^3 \frac{(9-z^2)z}{6}\ dz \\
&= \int_0^3 \frac{1}{6}(9z - z^3)\ dz \\
&= \frac{1}{6}(\frac{9}{2}z^2 - \frac{1}{4}z^4)|_0^3 \\
&= \frac{1}{6}(\frac{81}{2} - \frac{81}{4}) \\
&= \frac{1}{6}(\frac{162}{4} - \frac{81}{4}) \\
&= \frac{27}{8}.
\end{aligned}$$

Integrate the function $f(x, y, z) = x$ over the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$.

The tetrahedron is graphed below:



Note that we can set the integral up in any order we choose. Let's first integrate with respect to y; passing lines through the region parallel to the y axis, it is clear that all of the lines start on the angled plane and end at the plane $y = 1$:

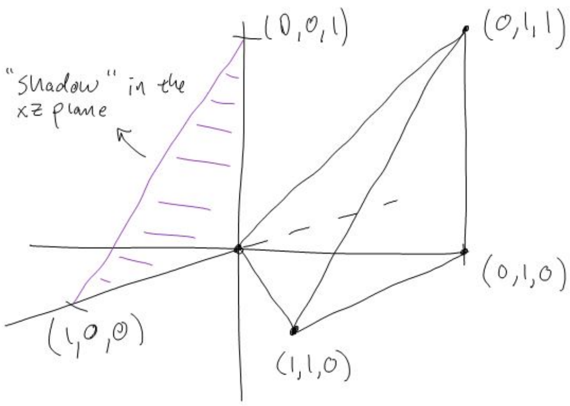


We need to determine a formula for the angled plane; to do so, we need to find a vector normal to the plane. Since the vectors $\langle 1,1,0\rangle$ and $\langle 0,1,1\rangle$ lie on the plane, we can take their cross product to get the desired normal vector. The cross product of the two vectors is

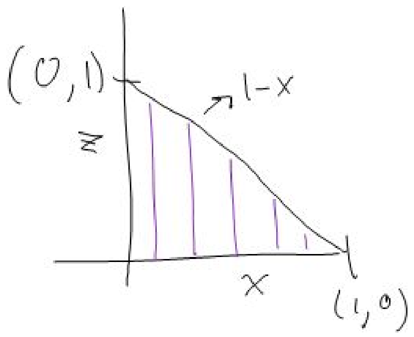
$$\langle 1,1,0\rangle \times \langle 0,1,1\rangle = \vec{i} - \vec{j} + \vec{k}.$$

Using $(0,0,0)$ as the chosen point on the plane and $\langle 1,-1,1\rangle$ as the normal vector, the equation for the plane is $x - y + z = 0$; we want this equation to give y as a function of x and z, so we write $y = x + z$. Thus the lower bound on y is $x + z$ and the upper bound is 1.

Now we need to determine the bounds on x and z; to do so, we need to look at the "shadow" of the tetrahedron in the xz plane. In a sense, we collapse the y coordinates, so that we see the triangle in the plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$:



Let's find the bounds on z in terms of x. The triangle has sides $x = 0$, $z = 0$, and $z = 1 - x$:



So the bounds on z are 0 and $1-x$; x varies from 0 to 1.
Thus the integral we want to evaluate is

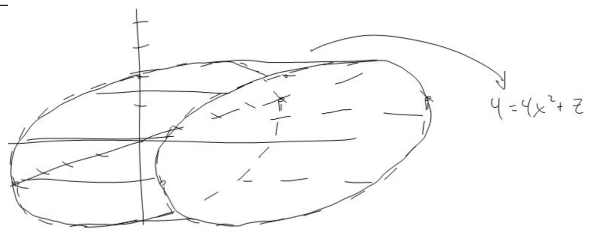
$$\begin{aligned}
\int_0^1\int_0^{1-x}\int_{x+z}^1 x\ dydzdx &= \int_0^1\int_0^{1-x} xy\big|_{x+z}^1\ dzdx \\
&= \int_0^1\int_0^{1-x} x - x(x+z)\ dzdx \\
&= \int_0^1\int_0^{1-x} x - x^2 - xz\ dzdx \\
&= \int_0^1\left((xz - x^2z - \frac{1}{2}xz^2)\big|_0^{1-x}\right)\ dx \\
&= \int_0^1 x(1-x) - x^2(1-x) - \frac{1}{2}x(1-x)^2\ dx \\
&= \int_0^1 x - x^2 - x^2 + x^3 - \frac{1}{2}x + x^2 - \frac{1}{2}x^3\ dx \\
&= \int_0^1 \frac{1}{2}x - x^2 + \frac{1}{2}x^3\ dx \\
&= (\frac{1}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4)\big|_0^1 \\
&= \frac{1}{4} - \frac{1}{3} + \frac{1}{8} \\
&= \frac{1}{24}.
\end{aligned}$$

Set up an integral that will yield the volume of the region in the first octant bounded between the cylinder $4 = 4x^2 + z^2$ and the plane $y = z + 2$.

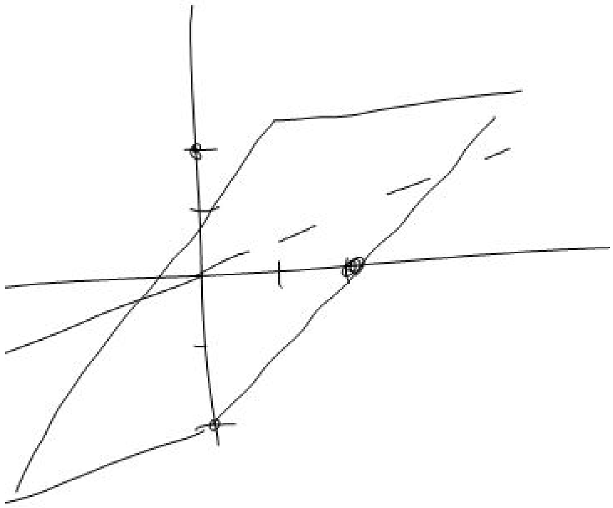
Recall that we can find the volume of a three dimensional region E by evaluating

$$\int\int\int_E 1dV.$$

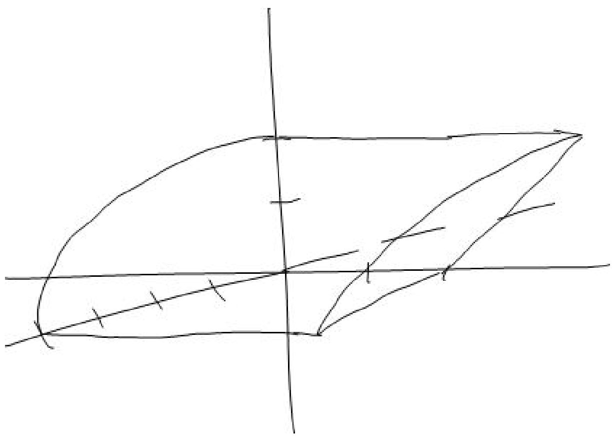
Let's start by sketching the region. The surface defined by $4 = 4x^2 + z^2$ is the elliptical cylinder graphed below:



The plane $y = z + 2$ is graphed below:

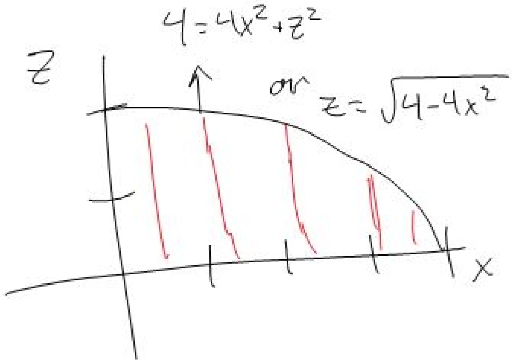


When the plane is graphed together with the cylinder, the shape cut out in space is graphed below:



Notice that if we pass lines through the shape parallel to the y axis, they always enter at $y = 0$ and exit through the plane $y = z + 2$. Thus it seems reasonable to integrate first with respect to y; it is clear that the bounds on y are 0 and $z + 2$.

Next, we should look at the "shadow" of the shape on the xz plane. We end up with the top quarter of an ellipse:



We may easily integrate in either direction; if we integrate first with respect to z (thinking of the top part of the ellipse as $z = \sqrt{4-4x^2}$), then the integral that yields the volume of the region is

$$\int_0^1 \int_0^{\sqrt{4-4x^2}} \int_0^{z+2} 1 \; dydzdx.$$