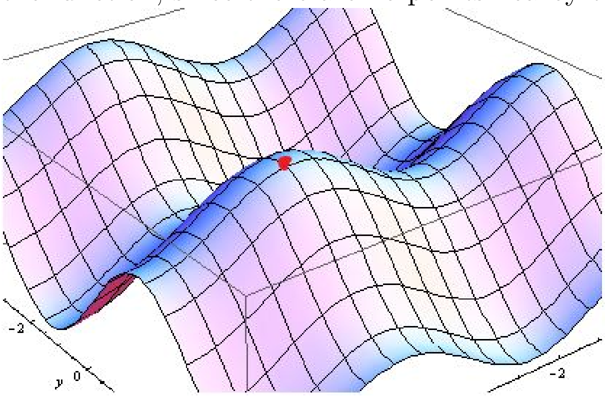
Extreme Values of Multivariate Functions

Our next task is to develop a method for determining local extremes of multivariate functions, as well as absolute extremes of multivariate functions on closed bounded regions. The ideas are quite similar to the corresponding problem for single variable functions.

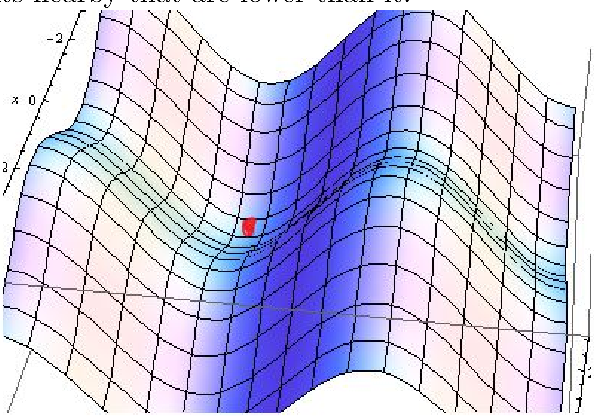
The concepts of "local maximum" and "local minimum" are fairly self-explanatory, but we give formal definitions here:

Definition 1. Let $f(x, y)$ be defined on a region R containing the point (a, b). Then $f(a, b)$ is a *local maximum* value of f if $f(a, b) \geq f(x, y)$ for all points (x, y) in the domain of f contained in a disk centered at (a, b). On the other hand, $f(a, b)$ is a *local minimum* value of f if $f(a, b) \leq f(x, y)$ for all points (x, y) in the domain of f contained in a disk centered at (a, b).

For example, the point marked in red on the surface $f(x, y) = \sin x + 2 \sin y$ is a local maximum for the function, since there are no points nearby that are higher than it:



Similarly, the point marked in red below is a local minimum for the function, since there are no points nearby that are lower than it:

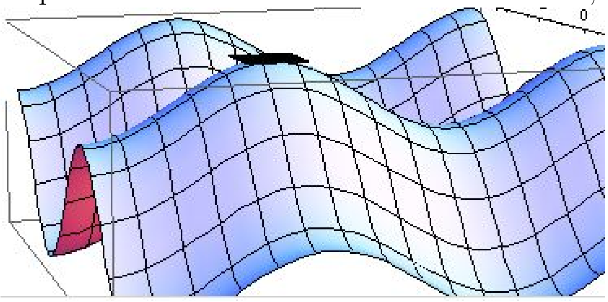


Recall that, in single-variable calculus, we were able to find candidates for local maxima and minima by noting that, if the function f is defined and differentiable at $x = a$, and has a local max or min at $x = a$, then $f'(a) = 0$. We have a similar test for multivariate functions:

Theorem 2. First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or local minimum value at a point (a, b) of its domain and if the first partial derivatives exist at (a, b), i.e. $f_x(a, b)$ and $f_y(a, b)$ exist, then $f_x(a, b) = f_y(a, b) = 0$.

Using the example above, notice that, were we to draw a tangent plane to the surface at either of the points that were indicated as local extrema, the plane would be parallel to the xy plane:

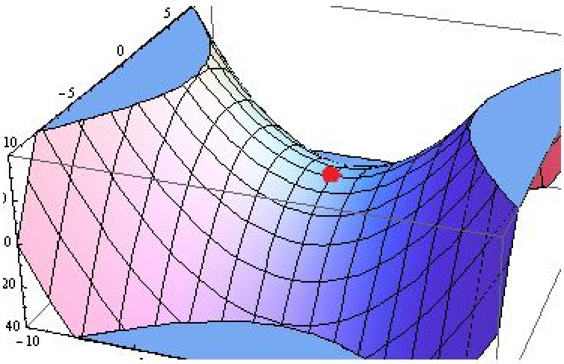


In other words, this particular plane must have an equation of the form $z = c$, where c is a constant. On the other hand, since we can calculate the tangent plane using the equation $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$, this indicates that $f_x = f_y = 0$ for the tangent plane at a local maximum or minimum.

If one of the partials f_x or f_y does not exist at (a, b), then the First Derivative Test is not applicable; so points where one or both of the partials do not exist could also give local maxima or minima. Accordingly, we define critical points of a function:

Definition 0.0.1. An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x or f_y does not exist is called a *critical point* of f.

Because of the first derivative test, the *only* points at which a function may have local extremes are at critical points. However, not all critical points actually occur at local minima; for example, consider the hyperbolic paraboloid $f(x, y) = x^2 - y^2$, which has $f_x(0, 0) = f_y(0, 0) = 0$:



It is clear that $(0, 0)$ does not yield a local maximum or a local minimum for f.

Definition 0.0.2. A *saddle point* (a, b) of a function $f(x, y)$ is a point that is a critical point, but yields neither a local maximum or a local minimum of $f(x, y)$. In other words, for any disk centered at (a, b), there is a point (x_1, y_1) so that $f(a, b) < f(x_1, y_1)$ and a point (x_2, y_2) so that $f(a, b) > f(x_2, y_2)$.

Since a critical point may actually be a saddle point, it is best to think of critical points as a list of all of the candidates for local extreme values of a function; we must test each critical point separately to determine if it does indeed yield a local maximum or minimum.

There are some cases where it will be quite simple to determine if a critical point does actually yield a local maximum or minimum. For example, if $f(x, y) = -x^2 - y^2$, then $f_x = -2x$ and

$f_y = -2y$. Then $f_x = f_y = 0$ only when $x = y = 0$, so that the only critical point is $(0,0)$. Since the function's value at this critical point is $f(0,0) = 0$, *and* the function is never positive, it is clear that this critical point yields a local maximum.

However, in most cases the analysis of critical points is not so simple. Fortunately, the following theorem provides a way to determine whether or not a critical point is actually a local extreme:

Theorem 3. Second Derivative Test for Local Extrema

Let $f(x,y)$ be a continuous function, all of whose first and second order partial derivatives are continuous throughout a disk centered at (a,b). If $f_x(a,b) = f_y(a,b) = 0$, then

1. f has a *local minimum* at (a,b) if $f_{xx}(a,b) > 0$ and $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$
2. f has a *local maximum* at (a,b) if $f_{xx}(a,b) < 0$ and $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$
3. f has a *saddle point* at (a,b) if $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) < 0$
4. The test is inconclusive at (a,b) if $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = 0$.

If the test is inconclusive, we will need to use some other method to determine if (a,b) is indeed a local maximum or minimum for f. In addition, the second derivative test does not help us deal with critical points where one of f_x or f_y do not exist. Again, in such a case we will need to use some other method to determine if the critical point yields a local extreme.

Examples

Find the local extremes of $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

The domain of f is the set of all (x,y) so that x and y are real numbers. We begin by determining all critical points of f, i.e. all points (a,b) in the domain of f where $f_x(a,b) = f_y(a,b) = 0$, or where one or both of the partials do not exist. The partials are given by

$$f_x(x,y) = -3x^2 + 4y \text{ and } f_y(x,y) = 4x - 4y.$$

The partials exist everywhere, so we only need to determine where they are simultaneously 0. If $f_y(x,y) = 0$, we must have $4x - 4y = 0$, i.e. $x = y$. In other words, every critical point must be of the form (x,x). So we may substitute $x = y$ into the equation for f_x; we have $-3x^2 + 4x = 0$. Factoring, we see that $x(-3x+4) = 0$, which means that $x = 0$ or $x = \frac{4}{3}$. Since every critical point looks like (x,x), we see that the only critical points are:

$$(0,0) \text{ and } \left(\frac{4}{3}, \frac{4}{3}\right).$$

We will now test these points using the Second Derivative Test.

To use the test, we will need to calculate $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$ for each of the critical points. Let's begin by finding the second order partials:

$$f_{xx}(x,y) = -6x, \; f_{yy}(x,y) = -4, \text{ and } f_{xy}(x,y) = 4.$$

Let's test the critical point $(0,0)$ first: since $f_{xx}(0,0)=0$, $f_{yy}(0,0)=-4$, and $f_{xy}(0,0)=4$, we have

$$f_{xx}(0,0)f_{yy}(0,0)-f_{xy}^2(0,0)=-16<0.$$

Looking back at conditions for the second derivative test, we see that $(0,0)$ can give neither a local maximum nor a local minimum, since $f_{xx}(0,0)=0$. However, since

$$f_{xx}(0,0)f_{yy}(0,0)-f_{xy}^2(0,0)<0,$$

$(0,0)$ agrees with the third condition, thus yields a saddle point.

Now we will test $\left(\frac{4}{3},\frac{4}{3}\right)$:

$$f_{xx}\left(\frac{4}{3},\frac{4}{3}\right)=-8,\ f_{yy}\left(\frac{4}{3},\frac{4}{3}\right)=-4,\ \text{and}\ f_{xy}\left(\frac{4}{3},\frac{4}{3}\right)=4.$$

So

$$f_{xx}\left(\frac{4}{3},\frac{4}{3}\right)f_{yy}\left(\frac{4}{3},\frac{4}{3}\right)-f_{xy}^2\left(\frac{4}{3},\frac{4}{3}\right)=32-16=16>0$$

Since $f_{xx}\left(\frac{4}{3},\frac{4}{3}\right)<0$ and $f_{xx}\left(\frac{4}{3},\frac{4}{3}\right)f_{yy}\left(\frac{4}{3},\frac{4}{3}\right)-f_{xy}^2\left(\frac{4}{3},\frac{4}{3}\right)>0$, $\left(\frac{4}{3},\frac{4}{3}\right)$ yields a local maximum.

Find the local extremes of $f(x,y)=x-\frac{1}{2}y^2-\frac{1}{3}x^3+y$.

Again, the domain of f is the set of all (x,y) so that x and y are real numbers.

We begin by locating the critical points for f: since

$$f_x(x,y)=1-x^2 \text{ and } f_y(x,y)=-y+1,$$

(both of which have domain all real numbers), we simply need to locate all points (x,y) that cause both partials to be 0. In order for $f_y=-y+1=0$, we must have $y=1$; so every critical point must be of the form $(x,1)$. If $f_x=1-x^2=0$, then either $x=1$ or $x=-1$, so the x coordinate of any critical point must either be 1 or -1. That means that the list of critical points is

$$(1,1) \text{ and } (-1,1).$$

Now to use the Second Derivative Test, we must calculate the second order partials:

$$f_{xx}(x,y)=-2x,\ f_{yy}(x,y)=-1,\ \text{and}\ f_{xy}(x,y)=0.$$

Next, we need to find $f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)$ for each of the critical points. Beginning with $(1,1)$, we have

$$f_{xx}(1,1)f_{yy}(1,1)-f_{xy}^2(1,1)=2.$$

Since $f_{xx}(1,1)<0$ and $f_{xx}(1,1)f_{yy}(1,1)-f_{xy}^2(1,1)>0$, the theorem says that f has a local maximum at $(1,1)$.

Finally, we check $(-1,1)$:

$$f_{xx}(-1,1)f_{yy}(-1,1)-f_{xy}^2(-1,1)=-2.$$

Since $f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) > 0$, the theorem says that f has a saddle point at $(-1,1)$.

Find the local extremes of $f(x,y) = 1 - (x^2+y^2)^{\frac{1}{3}}$.

The domain of f is the set of all (x,y) so that x and y are real numbers.
We need to find all critical points for f. The partial derivatives are

$$f_x(x,y) = -\frac{2x}{3(x^2+y^2)^{2/3}} \text{ and } f_y(x,y) = -\frac{2y}{3(x^2+y^2)^{2/3}}.$$

If $f_x(x,y) = 0$, then we must have $x = 0$, and if $f_y(x,y) = 0$, then $y = 0$. However, while the point $(0,0)$ is in the domain of $f(x,y)$, it is not in the domain of either $f_x(x,y)$ or $f_y(x,y)$, since $(0,0)$ creates a 0 in the denominator of each of f_x and f_y.

So while $(0,0)$ *is* a critical point of $f(x,y)$ (since the partials do not exist at the point), we can not test $(0,0)$ using the Second Derivative Test. Instead, we will need to carefully analyze the function: notice that $(x^2+y^2)^{\frac{1}{3}} > 0$ whenever one of x or y is nonzero. In particular, this means that

$$1 - (x^2+y^2)^{\frac{1}{3}} < 1 \text{ whenever } x \text{ or } y \text{ is nonzero.}$$

Since $f(0,0) = 1$, we see that this is indeed a local maximum for the function.

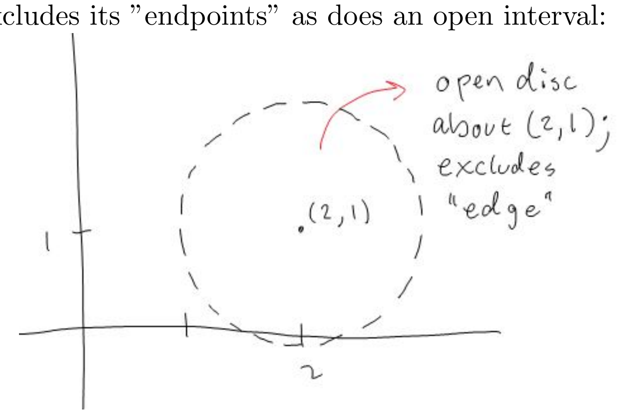
Open and Closed Sets

On the real number line, we think of an interval as being an "open interval" (more generally, an open set) if it does not include its endpoints; the interval $(3,5)$ is open. On the other hand, if the interval *does* include its endpoints, such as $[3,5]$, we say that it is a closed interval (or closed set). Finally, there are sets that are neither open nor closed, such as $(3,5]$ (since it includes one endpoint, but not the other).

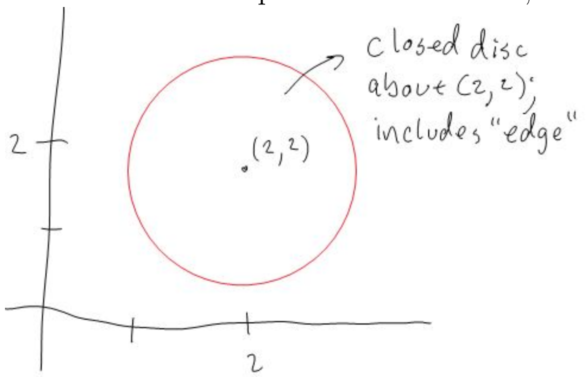
This idea of open and closed sets can be extended to sets in higher dimensions. We'll start with the two-dimensional case:

Definition 0.0.3. An *open disk of radius* r about the point (x,y) in the xy plane is the set of all points (x_0,y_0) so that $(x-x_0)^2 + (y-y_0)^2 < r^2$. The *closed disk of radius* r about the point (x,y) is the set of all points (x_0,y_0) so that $(x-x_0)^2 + (y-y_0)^2 \le r^2$.

An open disk is the two-dimensional analogue of an open set; in some sense, an open disk excludes its "endpoints" as does an open interval:



A closed disk corresponds to a closed set; the closed disk includes its "endpoints":



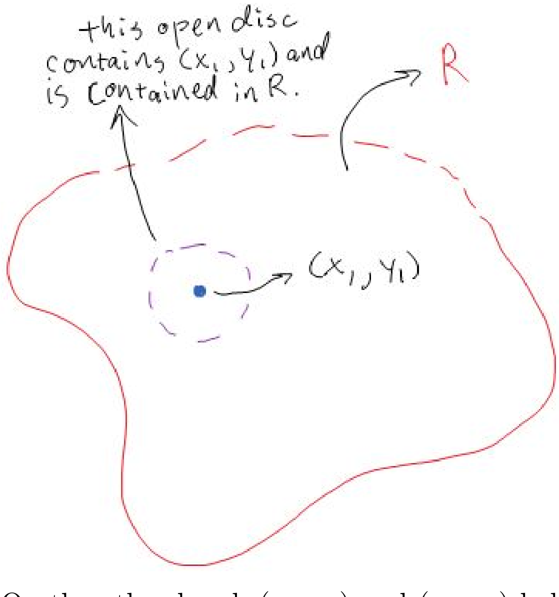
Given any region in two-dimensional space, we would like to be able to talk about it in terms of "open", "closed", or "neither". So we define:

Definition 0.0.4. A point (x, y) contained in a region R is called an *interior point* of R if there is an open disk centered at (x, y) that is contained completely in R.

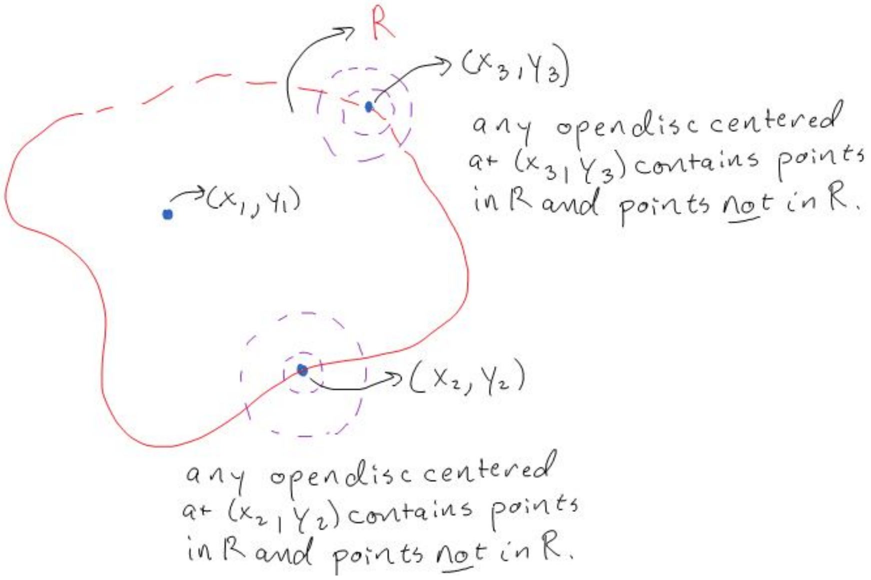
If the point (x, y) has the property that *any* open disk centered at (x, y) must contain a point of R other than (x, y) as well as a point not in R, we call (x, y) a *boundary point* of R.

By the definition, boundary points and interior points of a set R are distinct (no point can be both a boundary point and an interior point of R). Note that an interior point is required to be in the set R, but a boundary point is not. We will see that some boundary points may be in R, while others are not.

In the graph below, (x_1, y_1) is an interior point of R (as evidenced by the open disk containing it).



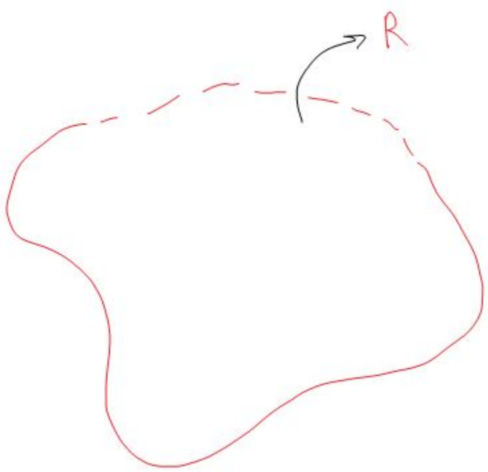
On the other hand, (x_2, y_2) and (x_3, y_3) below are boundary points of R; no matter how hard we try, we can not draw an open disk centered at (x_2, y_2) or at (x_3, y_3) that does not contain points inside as well as outside of R.



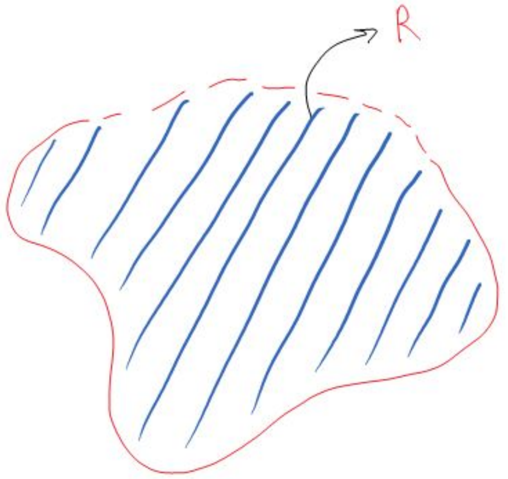
Note that (x_2, y_2) lies on the solid line, thus is included in R; however, (x_3, y_3) lies on the dotted line and is not included in R. Thus we see that a boundary point may be in the region R, or may not be included in R.

Definition 0.0.5. The set of all interior points of the region R is called the *interior* of R; the set of all boundary points of R is called the *boundary* of R. If every point of the region R is an interior point, we say that R is an *open set*. If the region R contains all of its boundary points, we say that R is a *closed set*.

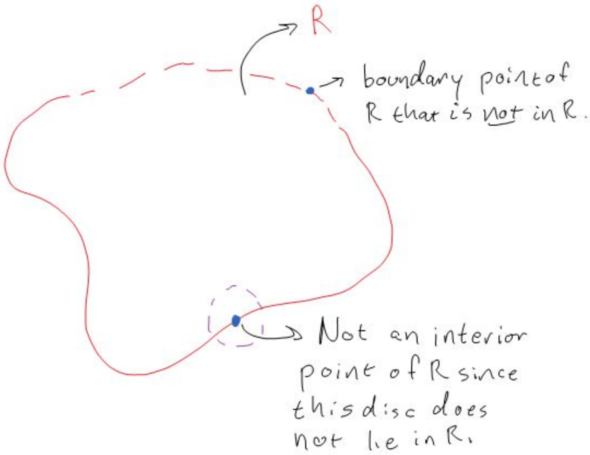
Examples:
Consider the region R graphed below:



The solid line indicates points in R, while the dotted line indicates points that are not in R.
The blue shaded region below is precisely the interior of R, and the red solid line and red dotted line make up the boundary of R:

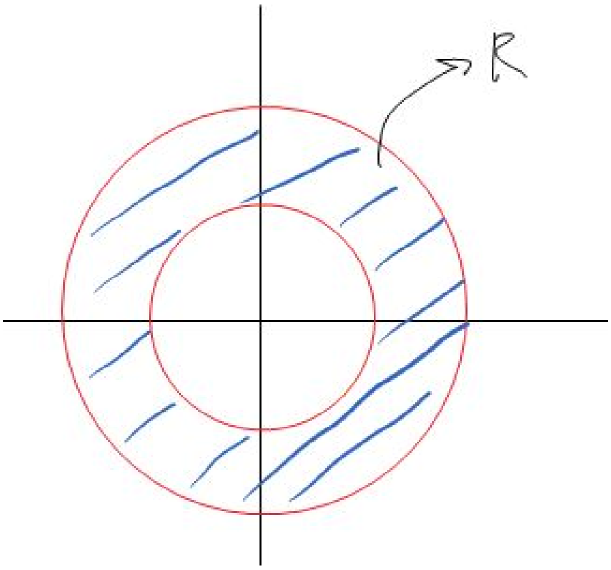


In the graph below, the point near the top is a boundary point of R but is not included in R, so R is not a closed set. On the other hand, R is not an open set since the point in R that is marked near the bottom is not an interior point:



As with the intervals we considered earlier, we see that a region can be neither open nor closed.

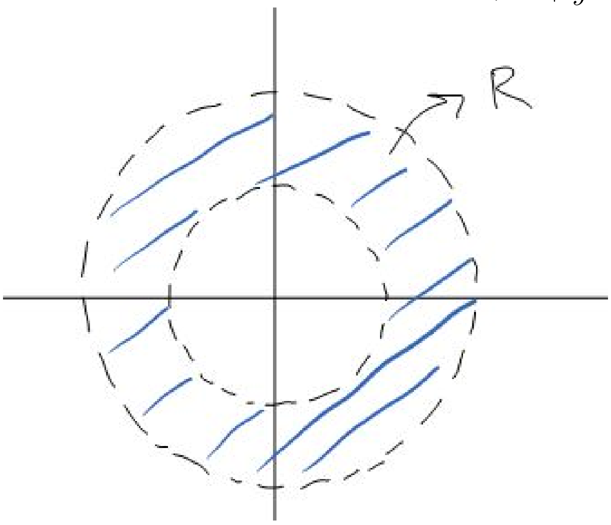
The region defined by $2 \leq x^2 + y^2 \leq 3$ is graphed below:



The interior points are shaded blue, and the boundary consists of the interior and exterior solid lines.

We can describe the boundary explicitly: (x_0, y_0) is a boundary point if $(x_0)^2 + (y_0)^2 = 2$ or if $(x_0)^2 + (y_0)^2 = 3$. Since all such points are in R, we see that R contains its entire boundary, thus is a closed set.

If we rewrite the definition of R as $2 < x^2 + y^2 < 3$, the graph of R is below:



The interior points are exactly the same as in the previous example, and the boundary is also the same. However, in this case the only points in R are the interior points. Thus R is an open region.

Finding Absolute Extrema of a Function on a Closed, Bounded Region

If we restrict the domain of the function $f(x, y)$ to a closed region R (where R is "relatively small"), it makes sense to discuss the "largest" and "smallest" values that f takes on.

Definition 0.0.6. Suppose that $f(x, y)$ is continuous on a closed bounded region R. Then:

1. An *absolute maximum* value of f is a point $f(x, y)$ so that $f(x, y) \geq f(x_0, y_0)$ for all points (x_0, y_0) in R.

2. An *absolute minimum* value of f is a point $f(x, y)$ so that $f(x, y) \leq f(x_0, y_0)$ for all points (x_0, y_0) in R.

Under these circumstances, f is guaranteed to take on absolute maximum and absolute minimum values somewhere on R. These values can occur either (1) on the interior of the region R, or (2) on the boundary of the region. We will need to test the two kinds of points separately.

Here are the steps we will follow to find the absolute extrema of f on the closed bounded region R:

(1) Draw the boundary of the region's domain in the xy plane.

(2) *Check the interior of* R: Find all critical points of $f(x, y)$. If (a, b) is a critical point of f that is also in R, we will test it in (4). Otherwise, discard it.

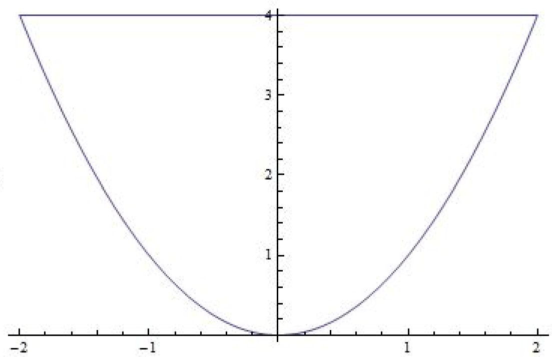
(3) *Check the boundary of* R: Write equations for the boundary of R, then find all local extrema of $f(x, y)$ that occur on this boundary; we will test these in (4). In particular, we will think of the boundary as a function of one variable, and will be able to use single-variable calculus to determine its local extremes. We must also test *all* endpoints of the boundary curves in (4), so include these in the list of possible absolute extrema.

(4) Test *all* points from (2) and (3) by plugging them into $f(x, y)$. The point (a_0, b_0) so that $f(a_0, b_0) \geq f(a, b)$ for any other point (a, b) on the list gives us the local maximum; the point (a_1, b_1) so that $f(a_1, b_1) \leq f(a, b)$ for any other point (a, b) on the list gives us the local minimum.

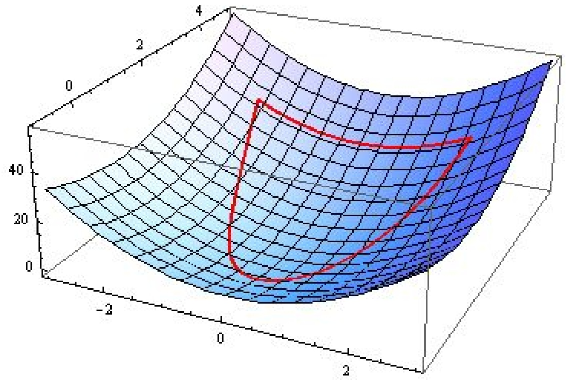
Examples:

Find the absolute extrema of the function $f(x, y) = 3x^2 + 2y^2 - 4y$ on the region in the xy plane bounded by $y = x^2$ and $y = 4$.

1. The domain of f is graphed below in the xy plane:



We need to find the absolute extremes of $f(x, y)$ that occur on and within the red boundary curves graphed below:



2. Now we need to find the critical points of $f(x, y)$ that occur in the domain of f; since $f_x(x, y) = 6x$ and $f_y(x, y) = 4y - 4$, the only point where f_x and f_y are simultaneously 0 is $(0, 1)$. Since this point *does* lie in the region graphed in (1), we will need to test it to see if it gives us an absolute extreme.

3. There are two "boundary curves" for the domain: $y = x^2$ and $y = 4$. We are interested in the behavior of f along each of these curves, so we consider the points on the surface defined by $f(x, y)$ so that $y = x^2$, and points on the surface so that $y = 4$.

   If $y = x^2$, then
   $$f(x, y) = f(x, x^2) = 3x^2 + 2x^4 - 4x^2 = 2x^4 - x^2.$$
   We may think of this as a function of one variable, $g(x) = 2x^4 - x^2$, and find its local extremes accordingly. Since $g'(x) = 8x^3 - 2x$, we can find the local extremes by setting $8x^3 - 2x = 0$ and factoring: if $2x(4x^2 - 1) = 0$ then either $x = 0$ or $x = \pm\frac{1}{2}$. These are the only possibilities for local extremes on $f(x, x^2)$, and these choices for x yield the points $(0, 0)$, $(\frac{1}{2}, \frac{1}{4})$, and $(-\frac{1}{2}, \frac{1}{4})$ (since $y = x^2$). We will test each of these points for absolute extrema in (4).

   Next, we need to check the curve $y = 4$; in this case, $f(x, 4) = 3x^2 + 32 - 16 = 3x^2 + 16$. Considering this curve as the function of one variable $h(x) = 3x^2 + 16$, we test h for local extremes. Since $h'(x) = 6x$, the only possibility is when $x = 0$. Returning to $f(x, y)$, we see that $(0, 4)$ is a possibility for an absolute extreme, and we will test it in the next step.

   Finally, the endpoints of the boundary curves could give us local extremes, so we will need to test $(-2, 4)$ and $(2, 4)$.

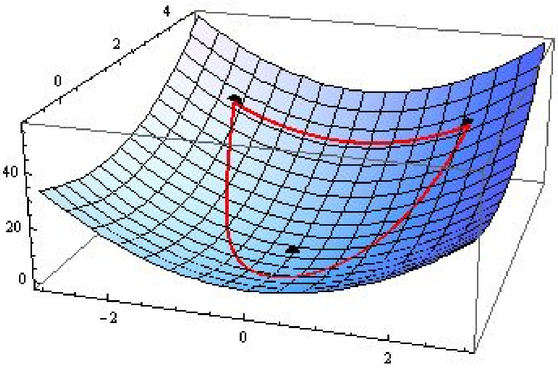
4. Here is the list of points that could potentially yield absolute extremes:
   $$(0, 1),\ (0, 0),\ (\frac{1}{2}, \frac{1}{4}),\ (-\frac{1}{2}, \frac{1}{4}),\ (0, 4),\ (-2, 4) \text{ and } (2, 4).$$
   Now we need to determine the value for $f(x, y)$ at each of the points:

$f(0,1) = -2$	$f(0,4) = 16$	$f(0,0) = 0$
$f(\frac{1}{2}, \frac{1}{4}) = -\frac{1}{8}$	$f(-\frac{1}{2}, \frac{1}{4}) = -\frac{1}{8}$	$f(-2,4) = 28$

$f(2,4) = 28$

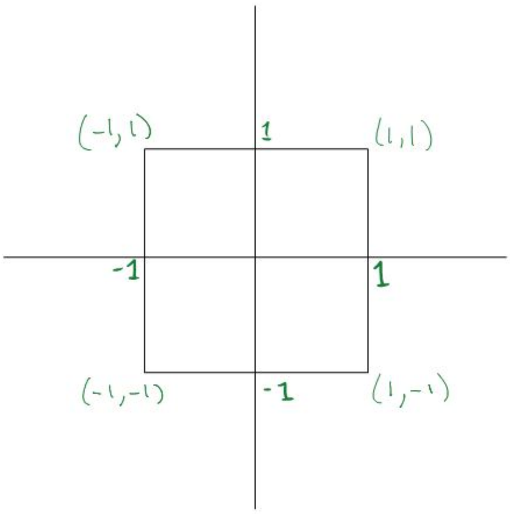
We see from the table that the function takes on maximum height at the points $(-2,4,28)$ and $(2,4,28)$, and minimum height at $(0,1,-2)$. So the maximum value for f is 28 and the minimum value is -2.

The function and the absolute extremes are plotted below:

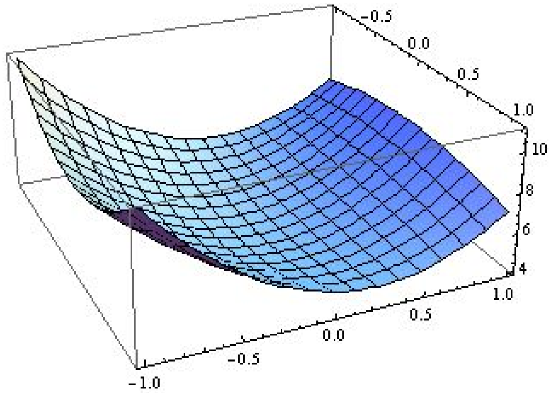


Find the absolute extreme values that $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ takes on when (x,y) are chosen so that $-1 \le x \le 1$ and $-1 \le y \le 1$.

1. The domain, a square in the xy plane, is graphed below:



The function is graphed below on its entire domain; now we need to find the absolute extremes in this region:



2. Next we need to find the critical points that lie in the domain. The partials are $f_x(x,y) = 2x - 4xy$ and $f_y(x,y) = 8y - 2x^2$; we need to determine what points (x,y) force both f_x and f_y to be 0. From the second equation, we see that f_y can be 0 only when $y = \frac{1}{4}x^2$. Using this information, we substitute $y = \frac{1}{4}x^2$ into the first equation, and we see that if $f_x = 0$ then $2x - 4x(\frac{1}{4}x^2) = 2x - x^3 = 0$. Factoring, we have $x(2 - x^2) = 0$, which means that either $x = 0$ or $x = \pm\sqrt{2}$. Since $y = \frac{1}{4}x^2$ (again, this was in order to make $f_y = 0$), we know that the critical points of f are $(0,0)$, $(\sqrt{2},1)$, and $(-\sqrt{2},1)$. However, the last two points are *not* in the rectangle above, since $\sqrt{2} > 1$. So the only point that we will need to test from this step is $(0,0)$.

3. Now we need to consider the boundary of the region. There are four boundary lines, given by $x = 1$, $x = -1$, $y = 1$, and $y = -1$. We must look for local extrema on each of these lines. Beginning with $x = 1$, the function $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ becomes $g(y) = 5 + 4y^2 - 2y$. Then $g'(y) = 8y - 2$. If $8y - 2 = 0$, then $y = \frac{1}{4}$. So we must test the point $(1, \frac{1}{4})$.

   Similarly, for $x = -1$, $f(x,y)$ becomes $h(y) = 5 + 4y^2 - 2y$; this is the same function as the one above, so its derivative is 0 when $y = \frac{1}{4}$; thus we will need to test the point $(-1, \frac{1}{4})$.

   For $y = 1$, $f(x,y)$ becomes $j(x) = -x^2 + 8$, and $j'(x) = -2x$. Since $j'(x) = 0$ when $x = 0$, we will need to test the point $(0,1)$.

   For $y = -1$, $f(x,y)$ becomes $j(x) = 3x^2 + 8$, and $j'(x) = 6x$. Again, $j'(x) = 0$ when $x = 0$, so we will need to test the point $(0,-1)$.

   Finally, we will need to test all the endpoints of the boundary curves, i.e. the corners $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$.

4. Here is the list of points that could potentially yield absolute extremes:

$$(0,0),\ (1,\frac{1}{4}),\ (-1,\frac{1}{4}),\ (0,1),\ (0,-1),\ (1,1),\ (1,-1),\ (-1,1) \text{ and } (-1,-1).$$

   Now we need to determine the value for $f(x,y)$ at each of the points:

$f(0,0) = 4$	$f(1,\frac{1}{4}) = 4.75$	$f(-1,\frac{1}{4}) = 4.75$
$f(0,1) = 8$	$f(0,-1) = 8$	$f(1,1) = 7$
$f(1,-1) = 11$	$f(-1,1) = 7$	$f(-1,-1) = 11$

Thus the absolute maximum of the function on the rectangle is 11, which occurs at the domain values $(1, -1)$ and $(-1, -1)$; and the absolute minimum value is 4, which occurs at the domain value $(0, 0)$.

The function and the absolute extremes are plotted below: