Multivariate Functions

In this chapter, we will return to scalar functions; thus the functions that we consider will output points in space as opposed to vectors. However, in contrast to the majority of functions with which we worked in previous calculus classes, our functions will accept multiple arguments.

A function $f(x) = y$ of one (independent) variable actually involves *two* variables, the input x and output y. Since there are two variables, we may graph the function in two dimensions. A two-variable function such as $f(x, y) = z$ involves two independent variables, x and y, and the dependent variable z. Thus a function of this form actually defines a surface in three dimensions. The total number of variables (both independent and dependent) of f dictates the number of physical dimensions needed to represent f; for example, the function $f(x, y, z) = w$ describes a hypersurface in 4 dimensional space. We think of the input of a multivariate function as an ordered list of numbers, which we call an n-tuple. For example, the list $(1, 0, -\pi, 3)$ is a 4-tuple.

In order to discuss multivariate functions, we must have definitions for domain, range, etc., analogous to the case of a function of one variable:

Definition 0.0.1. Let D be a collection of n-tuples of real numbers of the form $(x_1, x_2, \ldots, x_n)$. A *real-valued function* f on n variables is a rule that assigns a unique real number to each n-tuple from D; explicitly, we write

$$f(x_1, x_2, \ldots, x_n) = w.$$

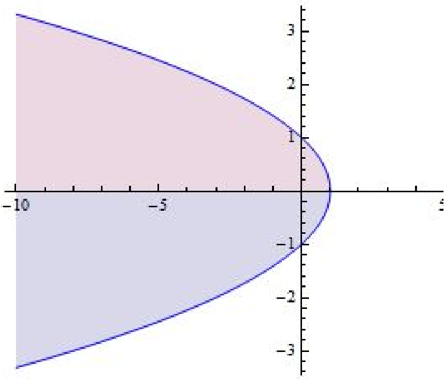
The set D is the *domain* of f, and the set of all w values that f outputs is the *range* of f. The number w is the dependent variable of f, and the numbers $x_1, x_2, \ldots, x_n$ are the independent variables.

For our purposes, when we are asked to calculate the domain of a function, we will assume that the domain is the largest possible collection of n-tuples that can be used as inputs for f.

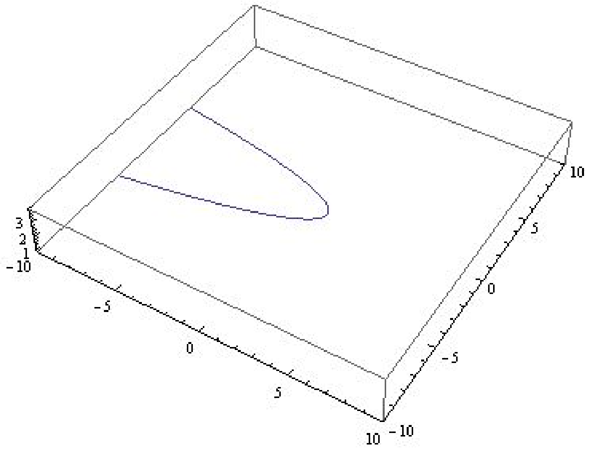
Examples:

Find the domain and range of the function $f(x, y) = \sqrt{1 - x - y^2}$.

To find the domain, we note that we must have $1 - x - y^2 \geq 0$. We can rewrite the inequality as $1 - x \geq y^2$ or $x \leq 1 - y^2$. The domain is the set of all points (x, y) so that $x \leq 1 - y^2$; this is the region shaded in below on the xy plane:



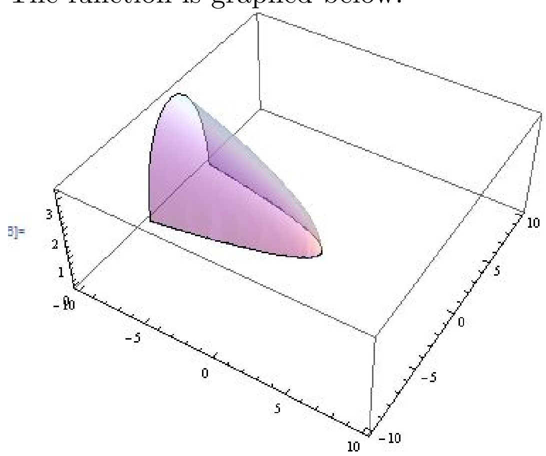
Let's embed the outline of the range in three dimensional space:



The graph of the surface should lie above (or below) this domain.

The range is the set of all possible outputs; we cannot force f to output negatives, but we can make $f(x, y) = 0$ by, for instance, choosing $x = 0$ and $y = -1$. However, there is no largest possible value for f, since we may (for example) fix $y = 0$ and choose x to be any negative number. So the range is $[0, \infty)$.

The function is graphed below:



Find the domain and range of the function $f(x, y, z) = \ln(25 - x^2 - y^2 - z^2)$.

We know that the natural log function only accepts non-negative inputs, so we must have $25 - x^2 - y^2 - z^2 > 0$. We can write the domain as $x^2 + y^2 + z^2 < 25$.

The largest possible value for the range occurs if we choose each of x, y, and z to be 0; $f(0, 0, 0) = \ln 25$. On the other hand, we can send f to $-\infty$ by, for example, choosing y and z to be 0, then allowing x to approach 5. Then $25 - x^2 - y^2 - z^2$ approaches 0, which sends the natural log function to $-\infty$. So the range is $(-\infty, \ln 25]$.

Find the domain and range of the function $f(x, y) = e^{\sqrt{x^2+y^2}}$.

Since x^2 and y^2 are always nonnegative, we may input any values we like for x and y into

$\sqrt{x^2+y^2}$. In addition, e^t has domain all reals, so $\sqrt{x^2+y^2}$ may take on any value we wish. So the domain of f is the entire xy plane.

To determine the range, notice that the smallest value for $\sqrt{x^2+y^2}$ is 0 (by choosing $x=y=0$); since e^t is increasing, $f(0,0)=0$ is the smallest possible value for f. On the other hand, by increasing x or y appropriately, we may make the outputs of f as large as we like. Thus f has domain $[0,\infty)$.

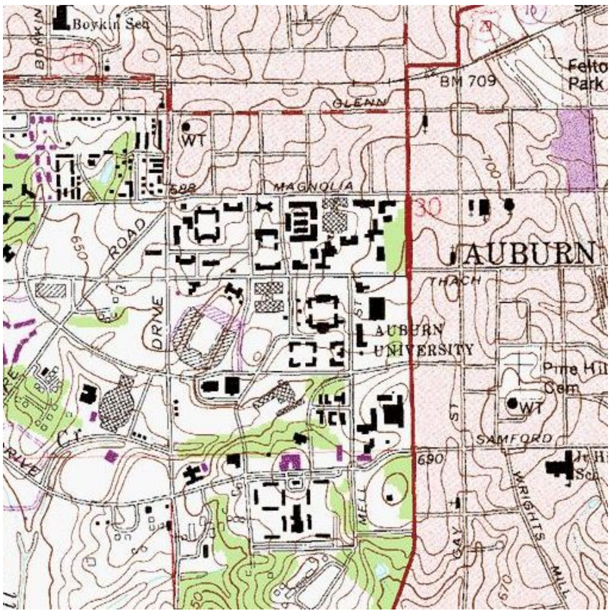
Graphing multivariate functions

For the obvious reasons, it is difficult to represent the graph of a function $f(x,y)$ of two variables in two dimensions, as $f(x,y)$ represents a surface in space. However, we can use the "level curves" of the surface to help us understand the three dimensional shape of the function's graph.

Definition 0.0.2. Given a function f of two variables and a constant c, a *level curve* at c of the function $f(x,y)$ is the set of all points (x,y) so that $f(x,y)=c$.

Level curves give us a way to represent a three-dimensional surface in two dimensions by "flattening" the surface, but retaining information about the function's height (z values).

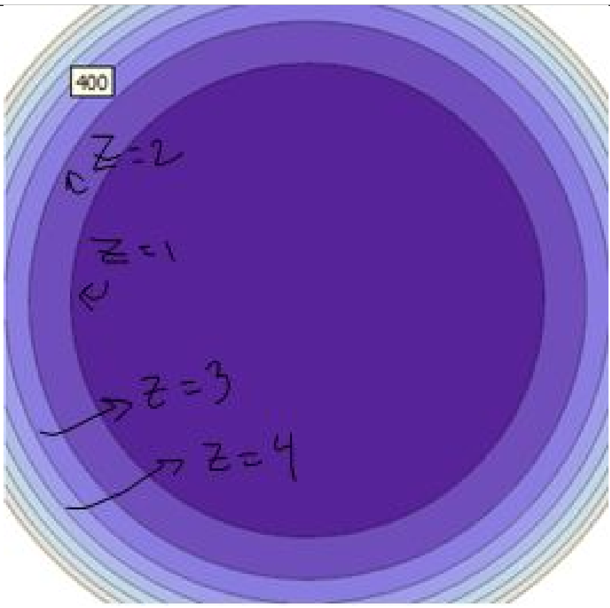
Topographical maps, such as the one below of Auburn's campus, use level curves to do precisely this; the lines on such a map represent points at the same height. If you were to pick a curve on the map and walk along the ground following this curve, you would stay at the same elevation the entire time.



We use the plot of the level curves of f to determine what the surface actually looks like. We can think of "stacking" the given curves at the right height, then connecting them to create the surface.

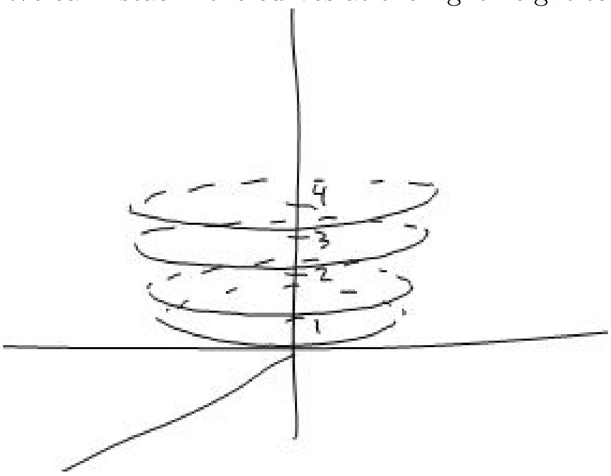
Examples

The graph below shows the level curves for a function $f(x,y)$:

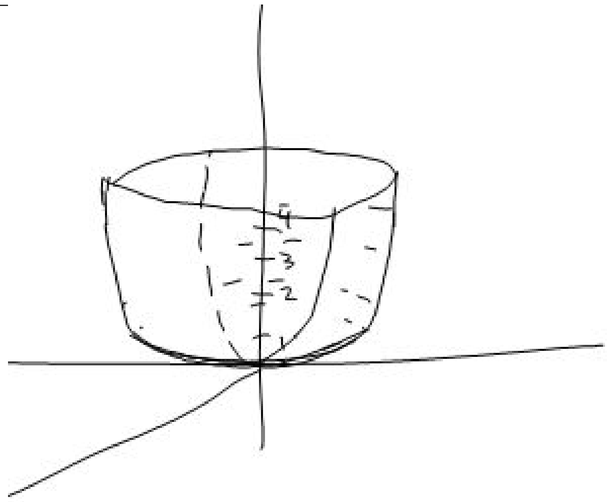


Use the level curves to draw a rough sketch of the surface.

We can "stack" the curves at the right height to get a rough idea of what the surface looks like:



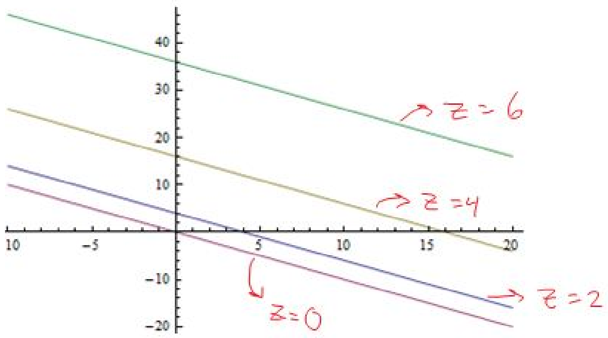
Connecting the curves gives us the surface below:



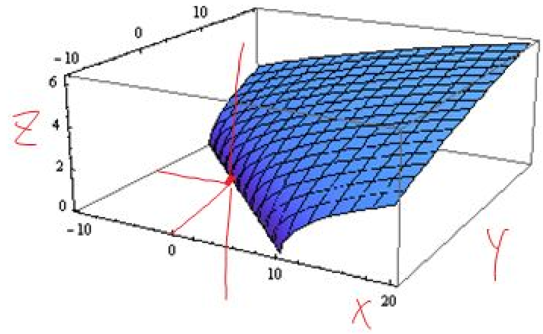
Find the domain and range of the function $f(x, y) = \sqrt{x + y}$. Graph 4 level curves of the function and use the level curves to sketch a rough graph of the surface.

The domain of $f(x, y)$ is the set of all points (x, y) so that $x + y > 0$, i.e. so that $-y < x$. The range is $[0, \infty)$.

The smallest value for which there is a level curve is $z = 0$, so we consider the curve given by $\sqrt{x + y} = 0$. This is the same as $x + y = 0$, i.e. $y = -x$, the negative of the identity line. Let's also sketch the level curves for $z = 2$, $z = 4$, and $z = 6$; we get the curves $y = 4 - x$, $y = 16 - x$, and $y = 36 - x$, respectively:



In this case, the level curves are lines, which leads us to sketch the surface as follows:

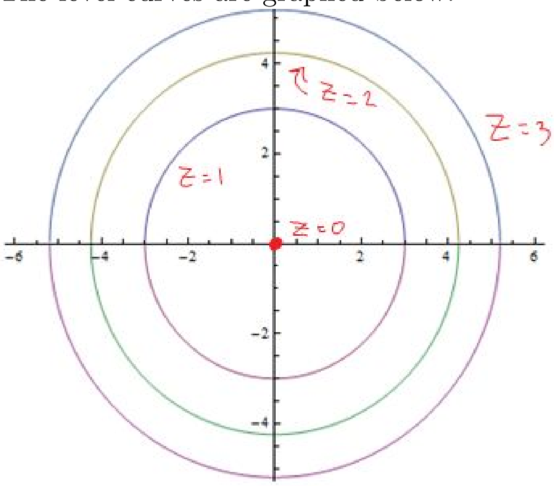


Find the domain and range of the function $f(x, y) = \dfrac{x^2}{9} + \dfrac{y^2}{9}$. Graph 4 level curves of the function and use the level curves to sketch a rough graph of the surface.

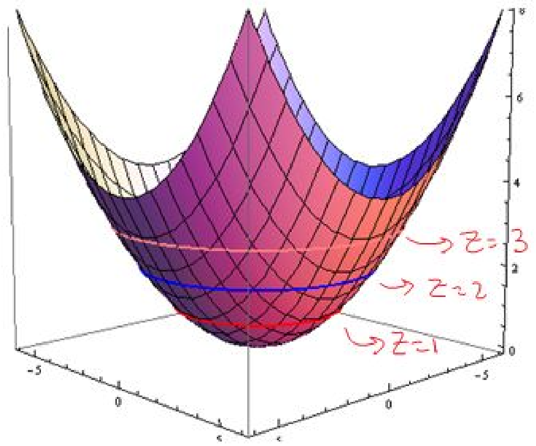
The domain is all pairs (x, y) of real numbers; the range is $[0, \infty)$. Again, the smallest level curve occurs when $z = 0$, so the corresponding curve is given by $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 0$; the only point satisfying this equation is $(x, y) = (0, 0)$.

The level curve at $z = 1$ is given by $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$ or $x^2 + y^2 = 9$, which is a circle of radius 3. The level curve at $z = 2$ is $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 2$ or $x^2 + y^2 = 18$, a circle of radius $\sqrt{18}$. The level curve at $z = 3$ is given by $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 3$ or $x^2 + y^2 = 27$, a circle of radius $\sqrt{27}$.

The level curves are graphed below:



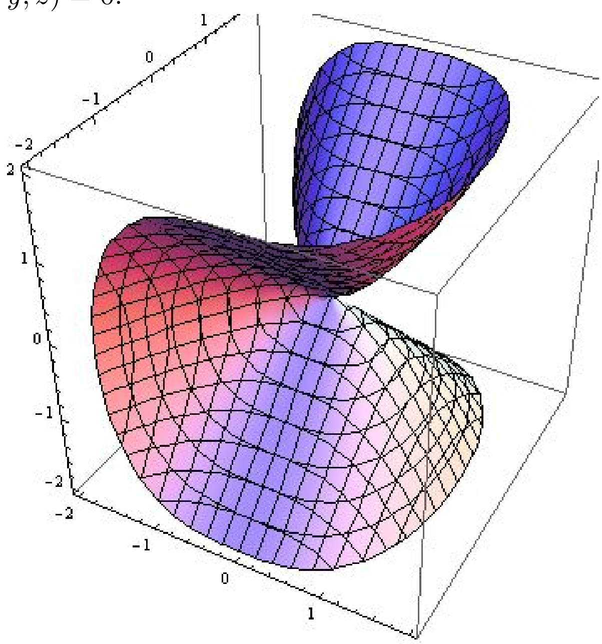
This leads to the surface in the graph below (the level curves are included on the graph):



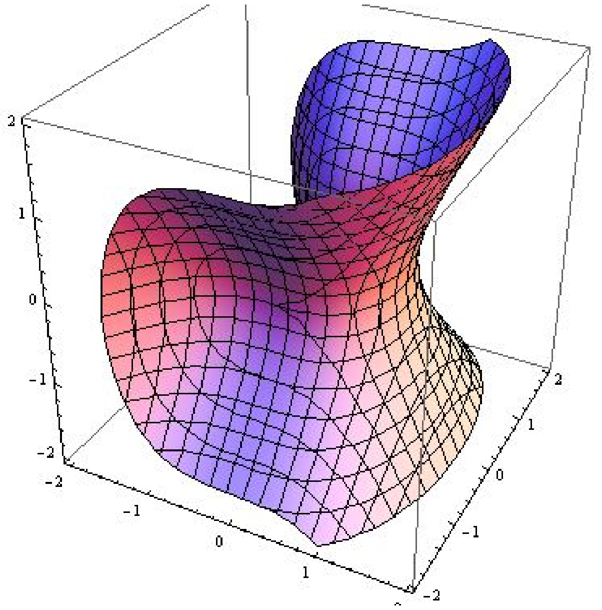
Level Surfaces

Just as a function $f(x, y)$ of two variables can be sliced into two-dimensional curves which we "stack" in three dimensions to get the surface that forms the graph of $f(x, y)$, a function $f(x, y, z)$ of three variables can be sliced into three dimensional surfaces which can be "stacked" in four dimensions to obtain the graph of $f(x, y, z)$. Obviously, we cannot actually visualize $f(x, y, z)$ since it is a hypersurface in 4 dimensional space, but thinking of the process as analogous to the 3 dimensional case can help us to get a better feel for what occurs here.

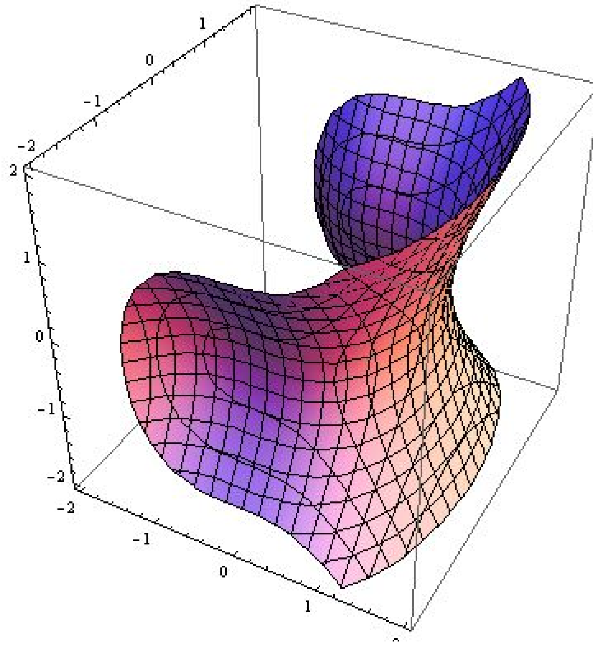
For example, if $f(x, y, z) = x^3 + y^2 - z^2$, then some of the level surfaces are graphed below. For $f(x, y, z) = 0$:



For $f(x, y, z) = 1$:



For $f(x, y, z) = 2$:



For $f(x, y, z) = 3$:

Each level surface is just one slice of the actual surface defined by $f(x, y, z)$. If we could "stack" the surfaces in 4 dimensions, we would be obtain the hypersurface $f(x, y, z)$.