Calculus on Vector Functions

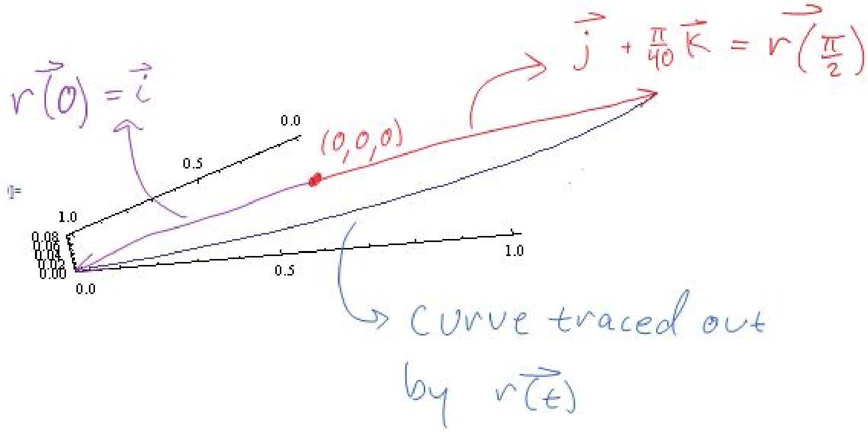
As mentioned in the previous section, calculus on vector functions is a completely different topic from the calculus of scalar valued functions. Although we can certainly discuss derivatives and integrals of vector functions, these terms have a slightly different shade of meaning than they do for scalar functions. In this section, we will establish the rules of calculus on vector functions.

Derivatives of Vector Functions

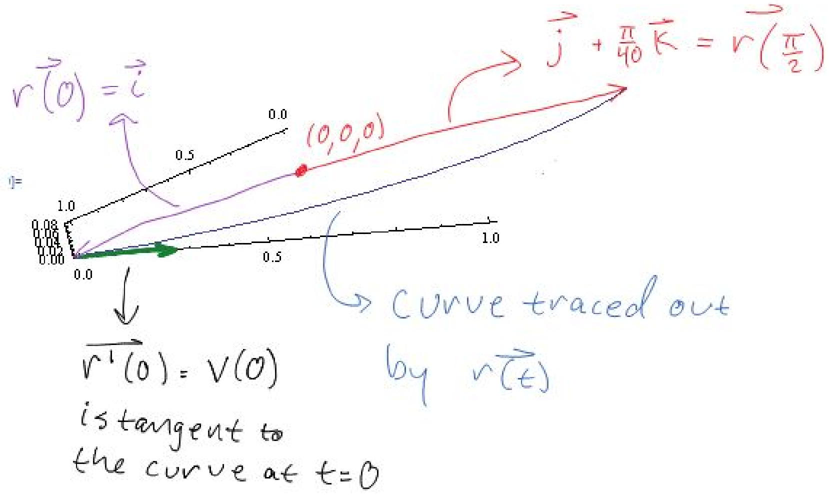
Derivatives of vector-valued functions are easy to evaluate. It only requires a straightforward calculation to show that, as with limits, derivatives of vector functions may be evaluated component by component.

However, before we look at a definition, let's think about the meaning of "derivative of a vector function". First of all, we know that the derivative of a function f is another function describing the rate of change of f.

In this particular case, we are considering a vector valued function $r(\vec{t})$; its derivative should also be a vector valued function, describing the rate of change of $r(\vec{t})$. Recall that, as t varies, the tip of $r(\vec{t})$ traces out a path in space:



In particular, $r'(\vec{t})$ is a vector tangent to the curve traced out by $r(\vec{t})$ at the parameter t:



Definition 0.0.1. The vector function

$$r(\vec{t}) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} = \langle f(t), g(t), h(t)\rangle$$

is **differentiable** at the point t if each of $f(t)$, $g(t)$ and $h(t)$ are, and the derivative $r'(\vec{t})$ is precisely

$$r'(\vec{t}) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} = \langle f'(t), g'(t), h'(t)\rangle.$$

We say that $r(\vec{t})$ is **differentiable** if it is differentiable at each point in its domain.

As mentioned above, we can think of $r'(\vec{t})$ as the function that gives *tangent vectors* to the curve traced out by $r(\vec{t})$. Since we often prefer unit vectors, we will define the *unit tangent vector* as the unit vector pointing in the same direction as $r'(\vec{t})$.

Definition 0.0.2. Given a differentiable vector function $r(\vec{t})$, the **unit tangent vector** of $r(\vec{t})$ is

$$\vec{T} = \frac{r'(\vec{t})}{|r'(\vec{t})|}.$$

Example
Given $r(\vec{t}) = t\vec{i} + \sin t\vec{j} + \cos t\vec{k}$, find $r'(\vec{t})$ and $\vec{T}$.

Differentiating components separately, we have

$$\begin{aligned} r'(\vec{t}) &= (\frac{d}{dt}t)\vec{i} + (\frac{d}{dt}\sin t)\vec{j} + (\frac{d}{dt}\cos t)\vec{k} \\ &= \vec{i} + \cos t\vec{j} - \sin t\vec{k}. \end{aligned}$$

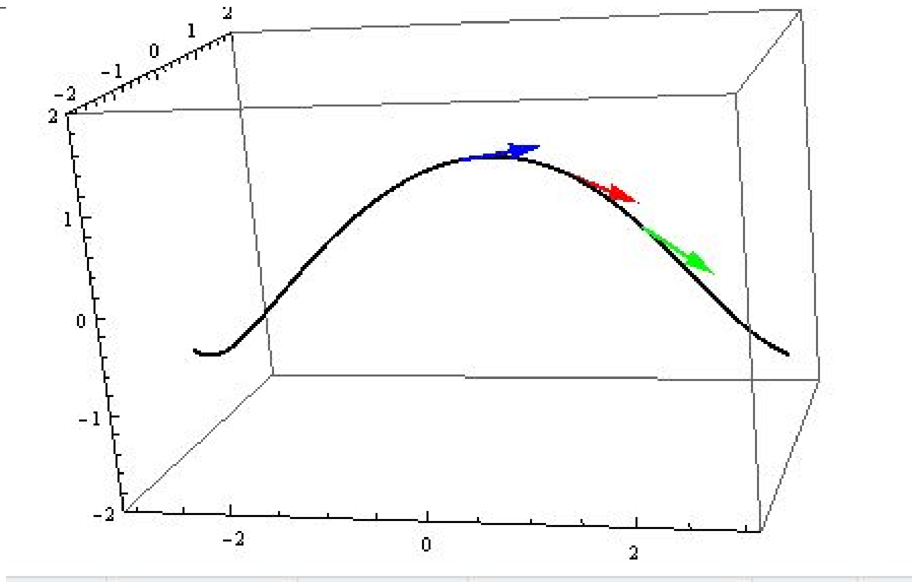
So the derivative is the *vector* function $r'(\vec{t}) = \vec{i} + \cos t\vec{j} - \sin t$.
Since $\vec{T} = \dfrac{r'(\vec{t})}{|r'(\vec{t})|}$, we need to calculate $|r'(\vec{t})|$:

$$\begin{aligned} |r'(\vec{t})| &= \sqrt{1 + \sin^2 t + \cos^2 t} \\ &= \sqrt{2}. \end{aligned}$$

So the unit tangent vector is given by

$$\vec{T} = \frac{1}{\sqrt{2}}\vec{i} + \frac{\cos t}{\sqrt{2}}\vec{j} - \frac{\sin t}{\sqrt{2}}\vec{k}.$$

The curve traced out by $\vec{rt}$ and three of its unit tangent vectors are graphed below:



We often use vector valued functions to describe the motion of some object in space; if the vector function $r(\vec{t})$ traces out the path C of a particle as $t \to \infty$, we call $r(\vec{t})$ the **position vector** for the particle.

Of course, derivatives have a special interpretation in this context. The derivative $r'(\vec{t})$ is a vector function tangent to the curve C. We write

$$v(\vec{t}) = r'(\vec{t})$$

and call $v(\vec{t})$ the particle's **velocity vector**. The velocity vector $\vec{v}$ always points in the direction of the motion of the particle. The vector $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector pointing in the direction of the motion of the particle. The magnitude (length) of the velocity vector, $|v(\vec{t})|$, is precisely the speed of the particle. We write speed$= |v(\vec{t})|$.

If the derivative of $v(\vec{t})$ exists, we write

$$a(\vec{t}) = \frac{d\vec{v}}{dt}$$

and call $a(\vec{t})$ the particle's **acceleration vector**.

Rules for Differentiating Vector Functions

Most of the rules for differentiating scalar functions have an analogue in the world of vector functions. In addition, since vector functions have two operations not available to scalar functions (cross product and dot product). we will need extra differentiation rules to describe how these vector operations interact with differentiation.

Theorem 3. Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t; let $\vec{C}$ be a constant vector, and c be a scalar. Let f be a differentiable scalar function.

1. Constant Rule $\frac{d}{dt}\vec{C} = \vec{0}$

2. Scalar Multiple Rules $\frac{d}{dt}(c\vec{u(t)}) = c\vec{u'(t)}$

   $\frac{d}{dt}(f(t)\vec{u(t)}) = f(t)\vec{u'(t)} + f'(t)\vec{u(t)}$

3. Sum/Difference Rule $\frac{d}{dt}(\vec{u(t)} \pm \vec{v(t)}) = \vec{u'(t)} \pm \vec{v'(t)}$

4. Dot Product Rule $\frac{d}{dt}(\vec{u(t)} \cdot \vec{v(t)}) = \vec{u'(t)} \cdot \vec{v(t)} + \vec{u(t)} \cdot \vec{v'(t)}$

5. Cross Product Rule $\frac{d}{dt}(\vec{u(t)} \times \vec{v(t)}) = \vec{u'(t)} \times \vec{v(t)} + \vec{u(t)} \times \vec{v'(t)}$

6. Chain Rule $\frac{d}{dt}\vec{u}(f(t)) = f'(t)\vec{u'}(f(t))$

Examples

With $\vec{r(t)} = t\vec{i} + \sin t\vec{j} + \cos t\vec{k}$ and $f(t) = 3t^3$, find $\frac{d}{dt}\vec{r}(f(t))$.

We've already calculated the derivative $\vec{r'(t)} = \vec{i} + \cos t\vec{j} - \sin t\vec{k}$. Since $f'(t) = 9t^2$, the derivative of $\vec{r}(f(t))$ is given by

$$\begin{aligned}\frac{d}{dt}\vec{r(f(t))} &= f'(t)\vec{r'}(f(t)) \\ &= 9t^2\vec{i} + 9t^2\cos(3t^3)\vec{j} - 9t^2\sin(3t^3)\vec{k}).\end{aligned}$$

Given the function $\vec{r(t)} = (2t\vec{i} + \sin t\vec{j}) \cdot (3t^2\vec{i} - \vec{k})$, find $\frac{dr}{dt}$.

By the dot product rule, we know that the derivative may be calculated as follows:

$$\begin{aligned}\frac{dr}{dt} &= \frac{d}{dt}(2t\vec{i} + \sin t\vec{j}) \cdot (3t^2\vec{i} - \vec{k}) + (2t\vec{i} + \sin t\vec{j}) \cdot (\frac{d}{dt}(3t^2\vec{i} - \vec{k})) \\ &= (2\vec{i} + \cos t\vec{j}) \cdot (3t^2\vec{i} - \vec{k}) + (2t\vec{i} + \sin t\vec{j}) \cdot (6t\vec{i}) \\ &= 6t^2 + 12t^2 \\ &= 18t^2.\end{aligned}$$

Alternatively, we could first evaluated the dot product, then differentiate. Since

$$\vec{r(t)} = (2t\vec{i} + \sin t\vec{j}) \cdot (3t^2\vec{i} - \vec{k}) = 6t^3,$$

we have

$$\frac{dr}{dt} = \frac{d}{dt}6t^3$$
$$= 18t^2.$$

Regardless of the order, we end up with the same *scalar* function $18t^2$.

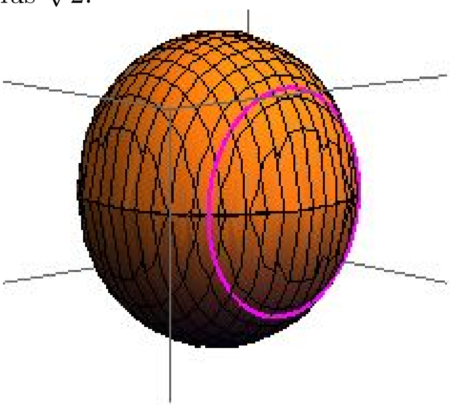
Vector Functions of Constant Length

As mentioned earlier, the derivative $r'(\vec{t})$ of the vector function $r(\vec{t})$ is a vector that is always tangent to the curve traced out by $r(\vec{t})$. In most cases, this does not say anything particularly special about the geometric relationship between $r'(\vec{t})$ and $r(\vec{t})$. However, there is one case in which the relationship is quite clear.

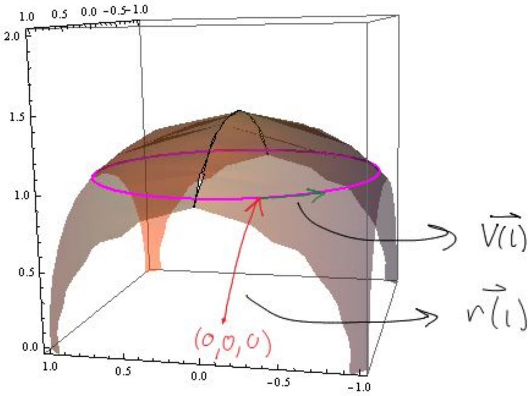
Suppose that the function $r(\vec{t})$ has constant length; if this is the case, then the curve traced by $r(\vec{t})$ lies on a sphere centered at the origin, with radius $|r(\vec{t})|$. For example, consider the function $r(\vec{t}) = \cos t\vec{i} + \vec{j} + \sin t\vec{k}$; the length of the vector is

$$|r(\vec{t})| = \sqrt{\cos^2 t + 1 + \sin^2 t}$$
$$= \sqrt{1+1}$$
$$= \sqrt{2}.$$

Since $r(\vec{t})$ has constant length $\sqrt{2}$, the curve traced out by the tip of $r(\vec{t})$ lies on a sphere of radius $\sqrt{2}$:



Now the vector $r'(\vec{t})$ is tangent to the curve as well as the sphere.



From the picture, it appears that $r(\vec{t})$ and $r'(\vec{t})$ are orthogonal; indeed it is quite simple to prove that this is true: suppose that $r(\vec{t}) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has constant length $|r(\vec{t})| = c$. Let's calculate $|r(\vec{t})|$ using the formula for length of a vector:

$$c = |r(\vec{t})| = \sqrt{(f(t))^2 + (g(t))^2 + (h(t))^2};$$

in particular,

$$c^2 = |r(\vec{t})|^2 = (f(t))^2 + (g(t))^2 + (h(t))^2.$$

However, note that $r(\vec{t}) \cdot r(\vec{t})$ is precisely

$$\begin{aligned} r(\vec{t}) \cdot r(\vec{t}) &= (f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}) \cdot (f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}) \\ &= (f(t))^2 + (g(t))^2 + (h(t))^2 \\ &= |r(\vec{t})|^2. \end{aligned}$$

So we have

$$\begin{aligned} \frac{d}{dt}(r(\vec{t}) \cdot r(\vec{t})) &= \frac{d}{dt}c^2 \\ (\frac{d}{dt}r(\vec{t})) \cdot r(\vec{t}) + r(\vec{t}) \cdot (\frac{d}{dt})r(\vec{t}) &= 0 \\ 2(\frac{d}{dt}r(\vec{t})) \cdot r(\vec{t}) &= 0 \\ r'(\vec{t}) \cdot r(\vec{t}) &= 0. \end{aligned}$$

Since a pair of vectors is orthogonal precisely when their dot product is 0, we see that $r(\vec{t})$ and its derivative $r'(\vec{t})$ are, in this case, orthogonal.

Note that this statement is *only* true when $r(\vec{t})$ has constant length; otherwise, $r(\vec{t})$ and $r(\vec{t})$ may not be orthogonal. We record the observation as a theorem:

Theorem 0.0.3. If $\vec{r(t)}$ is a differentiable vector function of constant length, then

$$\vec{r(t)} \cdot \vec{r'(t)} = 0,$$

that is $\vec{r(t)}$ and $\vec{r'(t)}$ are orthogonal.

Integrals of Vector Functions

We now define integration on vector functions. As with scalar functions, we first define antiderivatives and indefinite integrals:

Definition 0.0.4. If $\vec{R(t)}$ is a vector function so that $\frac{d}{dt}\vec{R(t)} = \vec{r(t)}$, we say that $\vec{R(t)}$ is an **antiderivative** of $\vec{r(t)}$. If $\vec{R(t)}$ is an antiderivative of $\vec{r(t)}$, then so is $\vec{R(t)} + \vec{C}$ for any constant vector $\vec{C}$. The **indefinite integral** of the vector function $\vec{r(t)}$ with respect to t is the set of all antiderivatives of $\vec{r(t)}$, denoted

$$\int \vec{r(t)}dt = \vec{R(t)} + \vec{C}.$$

Notice in particular that the constant $\vec{C}$ is a constant *vector*, thus is of the form $\vec{C} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ for some numbers c_1, c_2, and c_3.

Evaluating the definite integral of a vector function is as straightforward as we would expect; as with differentiating vector functions, we may integrate vector functions component by component.

Theorem 0.0.5. If $\vec{r(t)} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is a vector function so that each of the scalar functions $f(t)$, $g(t)$, and $h(t)$ is integrable over $[a, b]$, then $\vec{r(t)}$ is integrable over $[a, b]$ as well. The **definite integral** of $\vec{r(t)}$ over $[a, b]$ is

$$\int_a^b \vec{r(t)}dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}.$$

Using the definitions above, we have:

Theorem 0.0.6. The Fundamental Theorem of Calculus for Vector Functions

If the vector function $\vec{r(t)}$ is continuous on the interval $[a, b]$, and $\vec{R(t)}$ is any antiderivative of $\vec{r(t)}$, then

$$\int_a^b \vec{r(t)}dt = \vec{R(b)} - \vec{R(a)}.$$

Example

The acceleration vector of a particle is given by $\vec{a(t)} = (\sin t)\vec{i} - 2t\vec{k}$. The particle's velocity vector at time $t = 0$ is given by $\langle 3, -7, 2\rangle$ and its position at the same time is given by the vector $\langle 1, 0, 0\rangle$. Find the particle's velocity and position vectors.

Since $\vec{v(t)}$ is an antiderivative of $\vec{a(t)}$, we must integrate $\vec{a(t)}$ and use the initial conditions to determine the exact value for $\vec{v(t)}$.

By the previous theorem,

$$\begin{aligned} v(\vec{t}) &= \int a(\vec{t})dt \\ &= \int \big((\sin t)\vec{i} - 2t\vec{k}\big)dt \\ &= \Big(\big(\int \sin t dt\big)\vec{i} - \big(\int 2t dt\big)\vec{k}\Big)dt \\ &= -\cos t\vec{i} - t^2\vec{k} + \vec{C}. \end{aligned}$$

So $v(\vec{t}) = -\cos t\vec{i} - t^2\vec{k} + \vec{C}$. We need to determine the value for the vector $\vec{C}$. Since we know that $v(\vec{0}) = \langle 3, -7, 2\rangle$, we have

$$\begin{aligned} \langle 3, -7, 2\rangle &= v(\vec{0}) \\ &= (-\cos 0)\vec{i} - 0\vec{k} + \vec{C} \\ &= -\vec{i} + \vec{C}. \end{aligned}$$

Since $3\vec{i} - 7\vec{j} + 2\vec{k} = -\vec{i} + \vec{C}$, we conclude that $\vec{C} = 4\vec{i} - 7\vec{j} + 2\vec{k}$. So

$$\begin{aligned} v(\vec{t}) &= -\cos t\vec{i} - t^2\vec{k} + 4\vec{i} - 7\vec{j} + 2\vec{k} \\ &= (4 - \cos t)\vec{i} - 7\vec{j} + (2 - t^2)\vec{k}. \end{aligned}$$

We calculate $r(\vec{t})$ similarly:

$$\begin{aligned} r(\vec{t}) &= \int v(\vec{t})dt \\ &= \int \big((4 - \cos t)\vec{i} - 7\vec{j} + (2 - t^2)\vec{k}\big)dt \\ &= \big(\int 4 - \cos t dt\big)\vec{i} - \big(\int 7dt\big)\vec{j} + \big(\int 2 - t^2 dt\big)\vec{k} \\ &= (4t - \sin t)\vec{i} - 7t\vec{j} + (2t - \frac{1}{3}t^3)\vec{k} + \vec{D}. \end{aligned}$$

Since $r(\vec{0}) = \langle 1, 0, 0\rangle$, we may again determine the constant vector:

$$\begin{aligned} \langle 1, 0, 0\rangle &= r(\vec{0}) \\ &= (-\sin 0)\vec{i} - 0\vec{j} + 0\vec{k} + \vec{D} \\ &= \vec{D}. \end{aligned}$$

So in this case, $\vec{D} = \langle 1, 0, 0\rangle = \vec{i}$. Thus

$$\begin{aligned} r(\vec{t}) &= (4t - \sin t)\vec{i} - 7t\vec{j} + (2t - \frac{1}{3}t^3)\vec{k} + \vec{i} \\ &= (1 + 4t - \sin t)\vec{i} - 7t\vec{j} + (2t - \frac{1}{3}t^3)\vec{k}. \end{aligned}$$