Section 12.1

Three-Dimensional Coordinate Systems

The plane is a two-dimensional coordinate system in the sense that any point in the plane can be uniquely described using two coordinates (usually $x$ and $y$, but we have also seen polar coordinates $\rho$ and $\theta$). In this chapter, we will look at spaces with an extra dimension; in particular, a point in 3-space needs 3 coordinates to uniquely describe its location.

In the three-dimensional Cartesian coordinate system (or rectangular coordinate system), the coordinates $x$, $y$, and $z$ are measured against three mutually perpendicular axes.

The points $(2, 2, -1)$ and $(0, -1, 2)$ are graphed below in 3-space:

We determine the positive part of each axis using the right-hand rule; with the thumb of your right hand pointing upwards and your index finger perpendicular to your thumb, curl your remaining fingers in towards your palm; your index finger points in the direction of the positive part of the $x$ axis and your remaining fingers will curl through the positive parts of the $y$ axis; your thumb points in the direction of the positive part of the $z$ axis.

The axes define the three coordinate planes: the $xy$ plane is the set of all points so that $z = 0$ (and this equation defines the plane); the $yz$ plane is defined by the equation $x = 0$; likewise, the $xz$ plane is defined by $y = 0$. The three planes all intersect in one point, the origin (located at $(0, 0, 0)$), and divide 3 space into 8 octants (similar to the 4 quadrants in 2 dimensions). The octant in which all three coordinates are positive is called the first octant.
You can think of the room you’re sitting in as a three dimensional coordinate system; let the origin be the front left-hand bottom corner of the room. Then you are sitting in the first octant of the coordinate system.

**Examples**

Find the set of all points in 3 space satisfying \( x = y \) and \( z = 4 \).

The set of all points where \( z = 4 \) is the plane perpendicular to the \( xy \) plane at a height of 4:

Although all of the points on the plane satisfy \( z = 4 \), it is clear that many of the points do not satisfy \( x = y \), so we need to restrict our attention to the points graphed in red below:
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The set of points described by $x = y$ and $z = 4$ is the identity line in the plane parallel to the $xy$ plane.

Write a description for the set of all points in the plane passing through $(1, 1, 2)$ parallel to the $xz$ plane.

The point set is graphed below:

The $y$ value of any point on this plane must be 1, but the $x$ and $z$ coordinates may take on any values. So the equation $y = 1$ describes the plane.

Find the set of all points in 3 space satisfying $z \leq 0$, $x > y$.

Since $z \leq 0$, we can ignore the set of all points above the $xy$ plane. In 2-space, the set of points where $x > y$ is the set of points lying below the identity line:
In 3-space, the picture is similar; with the restriction that $z \leq 0$, the graph includes all of the points in the $xy$ plane so that $x > y$, as well as all of the points under this half-plane:

Most two-dimensional constructions have three-dimensional analogues. For instance, we can calculate the distance between a pair of points in 3-space using a nearly identical formula to what we used to calculate the distance between points in 2-space:

**Theorem 0.0.1.** The distance between a pair of points $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ is given by

$$D(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$ 

For example, the distance between $p_1 = (3, 1, 2)$ and $p_2 = (3, 5, -1)$ is

$$D(p_1, p_2) = \sqrt{(3 - 3)^2 + (5 - 1)^2 + (-1 - 2)^2} = \sqrt{16 + 0 + 9} = \sqrt{25} = 5.$$ 

So $p_1$ and $p_2$ are 5 units apart.

With $p_1$ as above, consider the set of all points $p_n$ that satisfy $D(p_1, p_n) = 5$, i.e. the set of all points that lie 5 units from $p_1$. This set forms the surface of a sphere centered at $p_1$.
The sphere has radius 5.

In general, the **standard equation for a sphere** centered at \( p_0 = (h, k, l) \) with radius \( r \) is given by

\[
(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.
\]

**Example:**

Show that the equation \( x^2 + y^2 + z^2 - 2x + 6z = 3 \) defines a sphere by putting it into standard form.

In order to put the equation in standard form, we will need to use the method of completing the squares.

**Reminder: Completing the Square**

To complete the square in \( x^2 + ax = c \) where \( a \) and \( c \) are constants, divide the coefficient \( a \) of \( x \) by 2 and square the resulting term to get \( \frac{a^2}{4} \). Then add \( \frac{a^2}{4} \) to both sides of the equation: we now have \( x^2 + ax + \frac{a^2}{4} = c + \frac{a^2}{4} \). This equation now has a perfect square on the left-hand side, which we rewrite as \( (x + \frac{a}{2})^2 = c + \frac{a^2}{4} \).

Returning to our original problem, we will complete the squares for each variable. Let’s start with the \( x \) terms:
The only term involving the variable \( y \) is already a perfect square, which we rewrite in standard form:

\[
(x - 1)^2 + (y - 0)^2 + z^2 + 6z = 4.
\]

Finally, we handle the \( z \)s:

\[
(x - 1)^2 + (y - 0)^2 + z^2 + 6z = 4
\]

\[
(x - 1)^2 + (y - 0)^2 + z^2 + 6z + 9 = 4 + 9
\]

\[
(x - 1)^2 + (y - 0)^2 + (z + 3)^2 = 13.
\]

The equation above is that of a sphere centered at \((1, 0, -3)\) with radius \(\sqrt{13}\).

Using the sphere in the example above,

1. The set of all points satisfying \((x - 1)^2 + (y - 0)^2 + (z + 3)^2 = 13\) is the surface of the sphere.
2. The set of all points satisfying \((x - 1)^2 + (y - 0)^2 + (z + 3)^2 < 13\) is the interior of the sphere.
3. The set of all points satisfying \((x - 1)^2 + (y - 0)^2 + (z + 3)^2 \geq 13\) is the surface and exterior of the sphere.