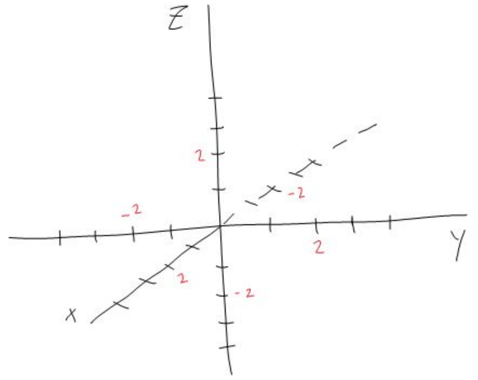
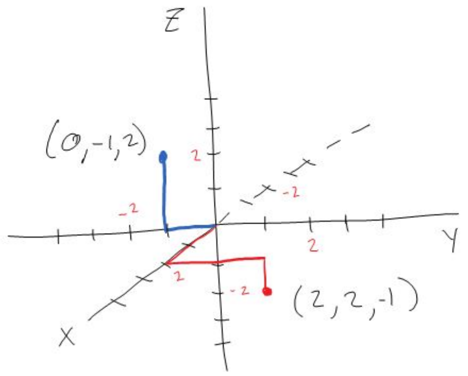
Three-Dimensional Coordinate Systems

The plane is a two-dimensional coordinate system in the sense that any point in the plane can be uniquely described using two coordinates (usually x and y, but we have also seen polar coordinates ρ and θ). In this chapter, we will look at spaces with an extra dimension; in particular, a point in 3-space needs 3 coordinates to uniquely describe its location.

In the three-dimensional Cartesian coordinate system (or rectangular coordinate system), the coordinates x, y, and z are measured against three mutually perpendicular axes.

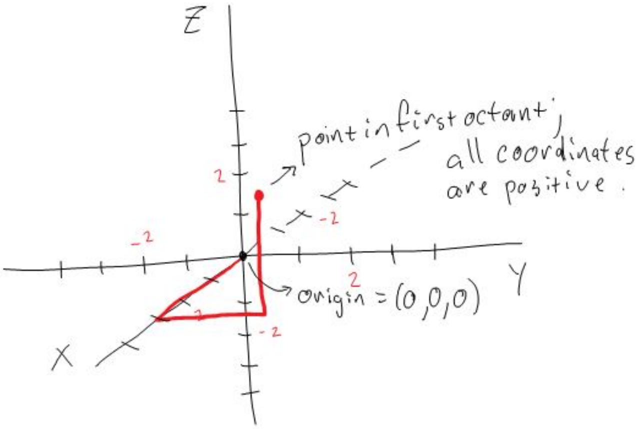


The points $(2, 2, -1)$ and $(0, -1, 2)$ are graphed below in 3-space:



We determine the positive part of each axis using the right-hand rule; with the thumb of your right hand pointing upwards and your index finger perpendicular to your thumb, curl your remaining fingers in towards your palm; your index finger points in the direction of the positive part of the x axis and your remaining fingers will curl through the positive parts of the y axis; your thumb points in the direction of the positive part of the z axis.

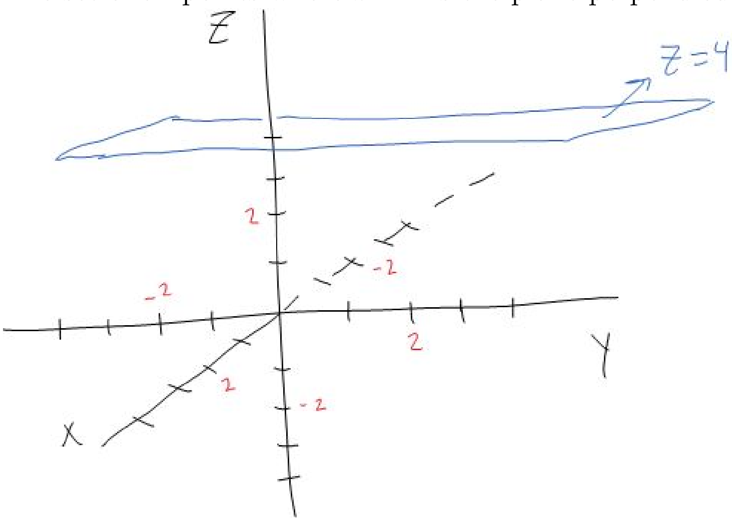
The axes define the three coordinate planes: the xy plane is the set of all points so that $z = 0$ (and this equation defines the plane); the yz plane is defined by the equation $x = 0$; likewise, the xz plane is defined by $y = 0$. The three planes all intersect in one point, the origin (located at $(0, 0, 0)$), and divide 3 space into 8 **octants** (similar to the 4 quadrants in 2 dimensions). The octant in which all three coordinates are positive is called the first octant.



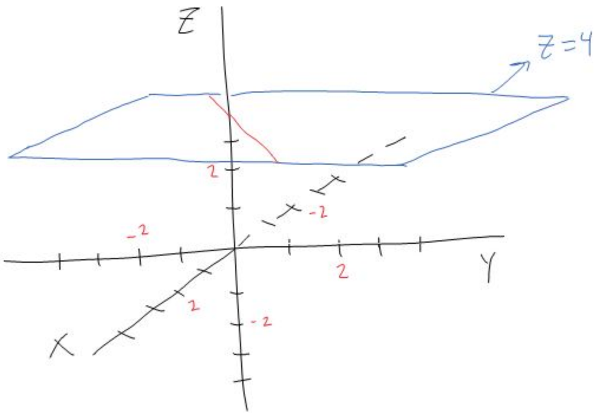
You can think of the room you're sitting in as a three dimensional coordinate system; let the origin be the front left-hand bottom corner of the room. Then you are sitting in the first octant of the coordinate system.

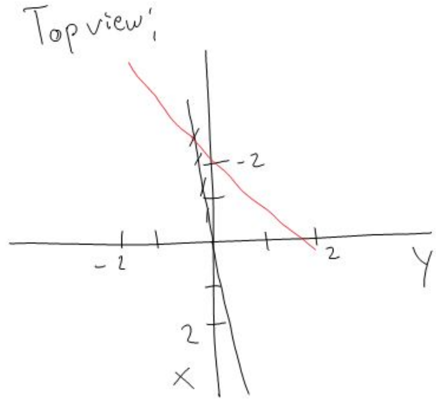
Examples
Find the set of all points in 3 space satisfying $x = y$ and $z = 4$.

The set of all points where $z = 4$ is the plane perpendicular to the xy plane at a height of 4:



Although all of the points on the plane satisfy $z = 4$, it is clear that many of the points do not satisfy $x = y$, so we need to restrict our attention to the points graphed in red below:

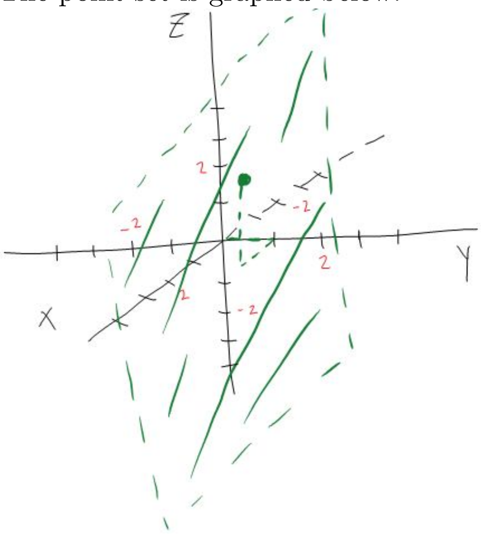




The set of points described by $x = y$ and $z = 4$ is the identity line in the plane parallel to the xy plane.

Write a description for the set of all points in the plane passing through $(1, 1, 2)$ parallel to the xz plane.

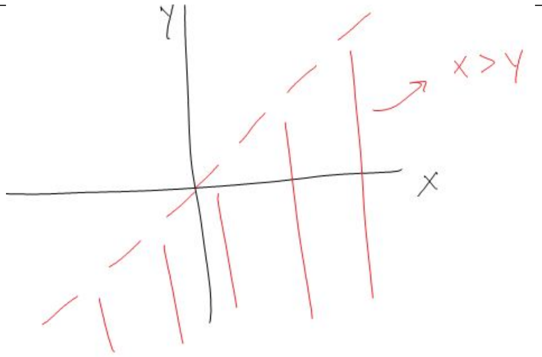
The point set is graphed below:



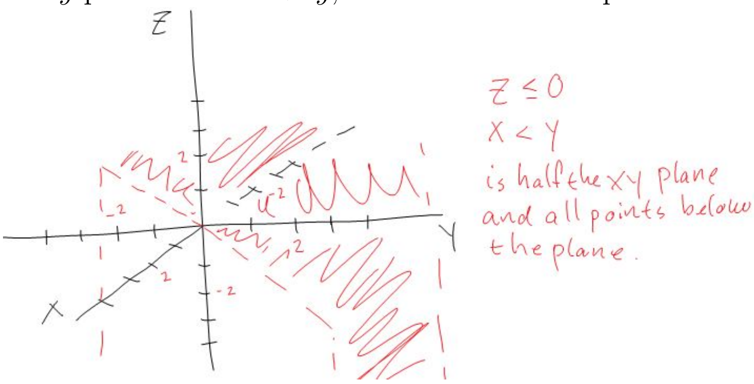
The y value of any point on this plane must be 1, but the x and z coordinates may take on any values. So the equation $y = 1$ describes the plane.

Find the set of all points in 3 space satisfying $z \leq 0$, $x > y$.

Since $z \leq 0$, we can ignore the set of all points above the xy plane. In 2-space, the set of points where $x > y$ is the set of points lying below the identity line:



In 3-space, the picture is similar; with the restriction that $z \leq 0$, the graph includes all of the points in the xy plane so that $x > y$, as well as all of the points under this half-plane:



Most two-dimensional constructions have three-dimensional analogues. For instance, we can calculate the distance between a pair of points in 3-space using a nearly identical formula to what we used to calculate the distance between points in 2-space:

Theorem 0.0.1. The distance between a pair of points $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ is given by

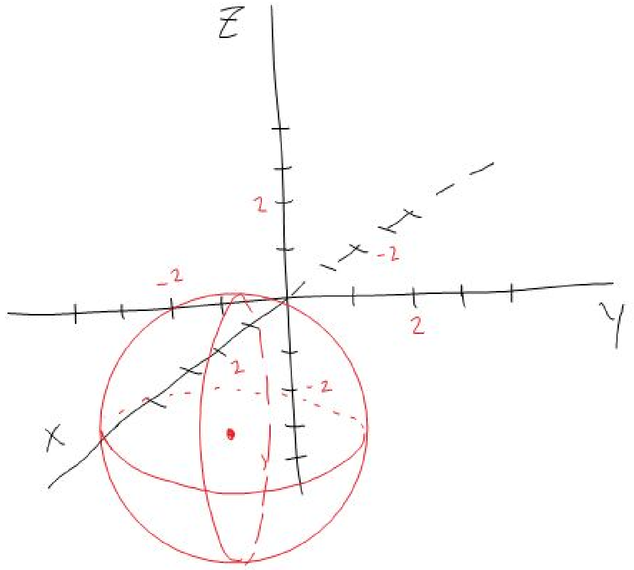
$$D(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

For example, the distance between $p_1 = (3, 1, 2)$ and $p_2 = (3, 5, -1)$ is

$$D(p_1, p_2) = \sqrt{(3-3)^2 + (5-1)^2 + (-1-2)^2} = \sqrt{16 + 0 + 9} = \sqrt{25} = 5.$$

So p_1 and p_2 are 5 units apart.

With p_1 as above, consider the set of *all* points p_n that satisfy $D(p_1, p_n) = 5$, i.e. the set of all points that lie 5 units from p_1. This set forms the surface of a sphere centered at p_1:



The sphere has radius 5.

In general, the **standard equation for a sphere** centered at $p_0 = (h, k, l)$ with radius r is given by

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

Example:

Show that the equation $x^2 + y^2 + z^2 - 2x + 6z = 3$ defines a sphere by putting it into standard form.

In order to put the equation in standard form, we will need to use the method of completing the squares.

Reminder: Completing the Square

To complete the square in $x^2 + ax = c$ where a and c are constants, divide the coefficient a of x by 2 and square the resulting term to get $\frac{a^2}{4}$. Then add $\frac{a^2}{4}$ to both sides of the equation: we now have $x^2 + ax + \frac{a^2}{4} = c + \frac{a^2}{4}$. This equation now has a perfect square on the left-hand side, which we rewrite as $(x + \frac{a}{2})^2 = c + \frac{a^2}{4}$.

Returning to our original problem, we will complete the squares for each variable. Let's start with the x terms:

$$\begin{aligned} x^2 + y^2 + z^2 - 2x + 6z &= 3 \\ x^2 - 2x + y^2 + z^2 + 6z &= 3 \\ x^2 - 2x + 1 + y^2 + z^2 + 6z &= 3 + 1 \\ (x-1)^2 + y^2 + z^2 + 6z &= 4. \end{aligned}$$

The only term involving the variable y is already a perfect square, which we rewrite in standard form:

$$(x-1)^2 + (y-0)^2 + z^2 + 6z = 4.$$

Finally, we handle the zs:

$$\begin{aligned} (x-1)^2 + (y-0)^2 + z^2 + 6z &= 4 \\ (x-1)^2 + (y-0)^2 + z^2 + 6z + 9 &= 4 + 9 \\ (x-1)^2 + (y-0)^2 + (z+3)^2 &= 13. \end{aligned}$$

The equation above is that of a sphere centered at $(1, 0, -3)$ with radius $\sqrt{13}$.

Using the sphere in the example above,

1. The set of all points satisfying $(x-1)^2 + (y-0)^2 + (z+3)^2 = 13$ is the surface of the sphere.
2. The set of all points satisfying $(x-1)^2 + (y-0)^2 + (z+3)^2 < 13$ is the interior of the sphere.
3. The set of all points satisfying $(x-1)^2 + (y-0)^2 + (z+3)^2 \geq 13$ is the surface and exterior of the sphere.