Math 162
Test 3
April 29, 2016

Name: ____________________________

You must show ALL of your work in order to receive credit.

If your scratch paper shows work that leads to your solution, please turn in inside the test. Otherwise, dispose of it yourself.
1. Given power series \( \sum_{n=1}^{\infty} \frac{(2x + 3)^n}{n!} \), find:

(a) The center of the series. (2 points)

\[ 2x + 3 = 0 \implies x = \frac{-3}{2} \]

(b) The interval of convergence of the series. (4 points)

**Ratio Test:**

\[
\lim_{n \to \infty} \frac{|2x + 3|^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{|2x + 3|}{n!} = 0 < 1
\]

So \( I = (-\infty, \infty) \)
2. Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n} \). (9 points)

Root test:
\[
\lim_{n \to \infty} \sqrt[n]{\frac{|x+1|^n}{n}} = \lim_{n \to \infty} \frac{|x+1|}{\sqrt[n]{n}} = |x+1|
\]

So converges if: \( |x+1| < 1 \)

\[-1 < x+1 < 1 \]

\[-2 < x < 0 \]

Test endpoints:

\( x = -2 \):
\[
\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]
Diverges - harmonic series.

\( x = 0 \):
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \]
Converges - alternating harmonic series.

So \( I = (-2, 0] \)
3. A power series centered at $a = 4$ converges at $x = 2$ and diverges at $x = 7$. For each of the following questions, answer true, false, or not enough information: (2 points each)

(a) The power series converges at $x = 5$.

\[ \text{True} \]

(b) The power series converges at $x = 9$.

\[ \text{False} \]

(c) The interval of convergence of the power series is $[2, 6]$.

\[ \text{Not enough information.} \]

4. Find the Taylor series centered at $a = 0$ generated by $f(x) = e^{-x^2}$. You may write your answer as a formula, or write down the first four non-zero terms of the series. (9 points)

\[ \text{Since } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

\[ e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \]
5. Find the Taylor series centered at \( a = \pi \) generated by \( f(x) = \cos(2x) \). You may write your answer as a formula, or write down the first three non-zero terms of the series. (9 points)

\[
\begin{align*}
f(x) &= \cos(2x) & f(\pi) &= 1 \\
f'(x) &= -2 \sin(2x) & f'(\pi) &= 0 \\
f''(x) &= -4 \cos(2x) & f''(\pi) &= -4 \\
f'''(x) &= 8 \sin(2x) & f'''(\pi) &= 0 \\
f^{(4)}(x) &= 16 \cos(2x) & f^{(4)}(\pi) &= 16
\end{align*}
\]

So the Taylor series generated by \( \cos(2x) \) at \( a = \pi \) is

\[
1 + \frac{0}{1!} (x-\pi) - \frac{4}{2!} (x-\pi)^2 + \frac{0}{3!} (x-\pi)^3 + \frac{16}{4!} (x-\pi)^4 + \ldots
\]

\[
= 1 - 2(x-\pi)^2 + \frac{2}{3} (x-\pi)^4 + \ldots
\]
\[
\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 3t
\]

6. Find the length of the curve given by parametric equations \(x(t) = t^3, y = 3t^2/2, 0 \leq t \leq \sqrt{3}\). (9 points)

\[
L = \int_{0}^{\sqrt{3}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{\sqrt{3}} \sqrt{3t^2 + 9t^4} \, dt
\]

\[
= \frac{\sqrt{3}}{2} \int_{0}^{\sqrt{3}} u^{1/2} \, du = \frac{3}{2} \left[ \frac{u^{3/2}}{3/2} \right]_{0}^{\sqrt{3}}
\]

\[
= (t^2 + 1)^{3/2} \bigg|_{0}^{\sqrt{3}} = (3 + 1)^{3/2} - 1 = 7
\]

7. Find the slope of the line tangent to the curve defined parametrically by \(x = \sin t + 1, y = 4e^{t/2}\) at the point (1, 4). (9 points)

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^{t/2}}{\cos t}
\]

at \(t = 0\), \(x = 1\), \(y = 4\), so \(\left. \frac{dy}{dx} \right|_{t=0} = \frac{2}{1} = 2\)
8. Find polar coordinates for the point in the xy plane whose Cartesian coordinates are $(2, 2\sqrt{3})$.

\[
\begin{align*}
\sqrt{r^2 + (2\sqrt{3})^2} &= 4 + 12 = 16 \\
\sqrt{r} &= 4
\end{align*}
\]

So $(4, \frac{\pi}{3})$ since $(2, 2\sqrt{3})$ is in Q1, \(\theta = \frac{\pi}{3}\)

9. Find alternate Polar coordinates for the point $(r, \theta) = (4, 2\pi/3)$ so that $r < 0$ and $0 \leq \theta \leq 2\pi$.

$(−4, \frac{5\pi}{3})$
10. The curve \( r = f(\theta) \) is graphed below in the \( r\theta \) plane. Graph the polar curve \( r = f(\theta) \) in the \( xy \) plane. Include appropriate scale markers on your graph. (9 points)
11. Set up an expression for the area enclosed by one loop of the curve \( r = \sin(2\theta) \). You do not have to evaluate the expression. (9 points)

\[
r = \sin 2\theta, \\
0 \leq \theta \leq \frac{\pi}{2}
\]

so

\[
A = \frac{1}{2} \int_{0}^{\pi/2} (\sin 2\theta)^2 d\theta
\]
12. Set up an expression for the area of the region that lies inside the curve $r = 1 - \cos \theta$ and outside the curve $r = 1 + \sin \theta$. You do not have to evaluate the expression. (9 points)

\[ \frac{1}{2} \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} (1 - \cos \theta)^2 - (1 + \sin \theta)^2 \, d\theta \]
13. Find a solution to the differential equation $y'' + y = 0$. (6 points)

$y = \cos x$ is a solution since

$y' = -\sin x$

$y'' = -\cos x$

and $y'' + y = -\cos x + \cos x = 0 \checkmark$