Advanced Integration Techniques: Trigonometric Integrals

We will use the following identities quite often in this section; you would do well to memorize them.

$$\begin{array}{ll} \sin^2 x = \frac{1-\cos(2x)}{2} & \cos^2 x = \frac{1+\cos(2x)}{2} \\ \cos(2x) = 1 - 2\sin^2 x & \cos(2x) = 2\cos^2 x - 1 \\ \sec^2 x = 1 + \tan^2 x & \csc^2 x = 1 + \cot^2 x \end{array} \qquad (1)$$

When attempting to integrate a function built up from trigonometric functions, there are often many different possibilities for choosing an integration technique. For example, we can solve $\int \sin x \cos x dx$ using the u-substitution $u = \cos x$. The same substitution could be used to find $\int \tan x dx$ if we note that $\tan x = \frac{\sin x}{\cos x}$. We can use integration by parts to solve $\int \sin(5x)\cos(3x)dx$. However, there are many other trigonometric functions whose integrals can not be evaluated so easily. In this section, we will look at multiple techniques for handling integrals of several different types of trig functions.

Integrals of the form $\int \sin^m x \cos^n x$

To integrate a function of the form $\int \sin^m x \cos^n x$, we will use one of the two following methods:

1. if both the powers m and n are even, rewrite both trig functions using the identities in (1)
2. Otherwise, we will rewrite the function so that only one power of $\sin x$ (or one power of $\cos x$) appears; this will allow us to make a helpful substitution:
   (a) If $m = 2k + 1$ is odd, then rewrite
   $$\sin^m x = \sin^{2k+1} x = (\sin x)(\sin^{2k} x) = (\sin x)(\sin^2 x)^k = (\sin x)(1 - \cos^2 x)^k,$$
   then use the u-substitution $u = \cos x$.
   (b) If $n = 2k + 1$ is odd, then rewrite
   $$\cos^n x = \cos^{2k+1} x = (\cos x)(\cos^{2k} x) = (\cos x)(\cos^2 x)^k = (\cos x)(1 - \sin^2 x)^k,$$
   then use the u-substitution $u = \sin x$.

Examples:

Find $\int \cos^3(2x)dx$.

Since $\cos(2x)$ has an odd power, let's rewrite

$$\cos^3(2x) = \cos(2x)\cos^2(2x) = \cos(2x)(1 - \sin^2(2x)).$$

Then

$$\int \cos^3(2x)dx = \int \cos(2x)(1 - \sin^2(2x))dx.$$

We will need the substitution $u = \sin(2x)$ so that $du = 2\cos(2x)dx$. Now we can finish the problem:

$$\begin{aligned}
\int \cos^3(2x)dx &= \int \cos(2x)(1-\sin^2(2x))dx \\
&= \frac{1}{2}\int 1 - u^2 du && \text{using the substitution } u = \sin(2x) \\
&= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C \\
&= \frac{1}{2}u - \frac{1}{6}u^3 + C \\
&= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C.
\end{aligned}$$

Find $\int \sin^3 x \cos^5 x dx$.

Since both trig functions have odd powers, we will rewrite one of them using the Pythagorean identity. Let's try

$$\begin{aligned}
\sin^3 x \cos^5 x &= \sin^3 x \cos^4 x \cos x \\
&= \sin^3 x (\cos^2 x)^2 \cos x \\
&= \sin^3 x (1 - \sin^2 x)^2 \cos x.
\end{aligned}$$

As in the previous example, we can use a simple u-substitution to finish the problem. Set $u = \sin x$ so that $du = \cos x dx$. Then

$$\begin{aligned}
\int \sin^3 x (1-\sin^2 x)^2 \cos x dx &= \int u^3(1-u^2)^2 du \\
&= \int u^3(1 - 2u^2 + u^4) du \\
&= \int u^3 - 2u^5 + u^7 du \\
&= \frac{1}{4}u^4 - \frac{2}{6}u^6 + \frac{1}{8}u^8 + C \\
&= \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C.
\end{aligned}$$

Find $\int \cos^2(2x)dx$.

Since there are no odd powers in this function, we will rewrite $\cos^2(2x) = \frac{1+\cos(4x)}{2}$ using the equation in (1). Then the integral calculation is fairly routine:

$$\begin{aligned}\int \cos^2(2x)dx &= \int \frac{1+\cos(4x)}{2}dx \\ &= \frac{1}{2}\int 1+\cos(4x)dx \\ &= \frac{1}{2}(x+\frac{1}{4}\sin(4x))+C \qquad \text{using the substitution } u=4x \\ &= \frac{1}{2}x+\frac{1}{8}\sin(4x)+C.\end{aligned}$$

Evaluate $\int \cos^2 x \sin^4 x dx$.

Since both the powers of $\cos x$ and $\sin x$ are even, we will write

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

and

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos(2x)}{2}\right)^2.$$

Then

$$\begin{aligned}\int \cos^2 x \sin^4 x dx &= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)^2 dx \\ &= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-2\cos(2x)+\cos^2(2x)}{4}\right)dx \\ &= \frac{1}{8}\int 1-2\cos(2x)+\cos^2(2x)+\cos(2x)-2\cos^2(2x)+\cos^3(2x)dx \\ &= \frac{1}{8}\int 1-\cos(2x)-\cos^2(2x)+\cos^3(2x)dx \\ &= \frac{1}{8}\left(x-\frac{1}{2}\sin 2x-\int \cos^2(2x)dx+\int \cos^3(2x)\right)dx.\end{aligned}$$

We have already showed that

$$\int \cos^2(2x)dx = \frac{1}{2}x+\frac{1}{8}\sin(4x)+C$$

and

$$\int \cos^3(2x)dx = \frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)+C,$$

so finally we have

$$\int \cos^2 x \sin^4 x dx = \frac{1}{8}\left(x-\frac{1}{2}\sin 2x-\frac{1}{2}x-\frac{1}{8}\sin(4x)+\frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)\right)+C.$$

Integrating powers of $\tan x$, $\sec x$, $\csc x$, and $\cot x$

To integrate powers of the other trig functions, we will often need to use u-substitution or integration by parts together with the pythagorean identities; if possible, we will need to take advantage of the fact that $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sec^2 x = \sec x \tan x$, $\frac{d}{dx}\csc x = -\csc x \cot x$, and $\frac{d}{dx}\cot x = -\csc^2 x$.

Example:

Evaluate $\int \csc^4 x dx$.

Writing $\csc^4 x = (\csc^2 x)(\csc^2 x) = (1+\cot^2 x)(\csc^2 x)$ is advantageous, as it will allow us to use the substitution $u = \cot x$:

$$\begin{aligned}
\int \csc^4 x dx &= \int (1+\cot^2 x)(\csc^2 x) dx \\
&= -\int (1+u^2) du \qquad \text{using } u = \cot x \text{ and } -du = \csc^2 x dx \\
&= -u - \frac{1}{3}u^3 + C \\
&= -\cot x - \frac{1}{3}\cot^3 x + C.
\end{aligned}$$

Eliminating square roots

If the function we wish to integrate involves the square root of some trigonometric function, we may be able to eliminate the root by using the pythagorean identities or the identities from (1).

Examples:

Evaluate $\int \sqrt{\cos y + 1} dy$

The identity $\cos^2 x = \frac{1+\cos(2x)}{2}$ can help us here. Setting $y = 2x$, so that $x = \frac{y}{2}$, the identity becomes $\cos^2(\frac{y}{2}) = \frac{1+\cos y}{2}$. We would like to replace the quantity $\cos y + 1$; solving for this quantity in the above identity, we have $\cos y + 1 = 2\cos^2(\frac{y}{2})$. So we may rewrite the integral as

$$\begin{aligned}
\int \sqrt{\cos(y)+1} dx &= \int \sqrt{2\cos^2\left(\frac{y}{2}\right)} dy \\
&= \int \sqrt{2}\cos\left(\frac{y}{2}\right) dy \\
&= 2\sqrt{2}\sin\left(\frac{y}{2}\right) + C.
\end{aligned}$$

Find $\int \sqrt{\csc^2\theta - 1} d\theta$.

Since $\csc^2\theta - 1 = \cot^2\theta$, let's rewrite

$$\begin{aligned}\int \sqrt{\csc^2\theta - 1}d\theta &= \int \sqrt{\cot^2\theta}d\theta \\ &= \int \cot\theta d\theta.\end{aligned}$$

Since $\cot\theta = \frac{\cos\theta}{\sin\theta}$, we can integrate the function using a substitution; setting $u = \sin\theta$ so that $du = \cos\theta d\theta$, we have

$$\begin{aligned}\int \sqrt{\csc^2\theta - 1}d\theta &= \int \cot\theta d\theta \\ &= \int \frac{\cos\theta}{\sin\theta}d\theta \\ &= \int \frac{1}{u}du \\ &= \ln u + C \\ &= \ln(\sin\theta) + C.\end{aligned}$$

A brief aside

We have not yet learned how to evaluate $\int \sec x dx$, and as we will need to know this integral in future sections, let's go ahead and compute it. It turns out that the best way to evaluate the integral is by using Mathemagic: note that $\sec x$ can be rewritten as

$$\sec x = \sec x \frac{\sec x + \tan x}{\sec x + \tan x} = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}.$$

This may seem pointless, but it will actually allow us to use a basic substitution to evaluate the integral. Setting $u = \sec x + \tan x$ so that $du = \sec x \tan x + \sec^2 x$, we have

$$\begin{aligned}\int \sec x dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx \\ &= \int \frac{1}{u}du \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$