Indeterminate Forms and L'Hopital's Rule

When we computed limits of quotients (such as $\lim_{x\to a} \frac{f(x)}{g(x)}$), we came across several different things that could happen:

1. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$; in particular, $\lim_{x\to a} g(x) \neq 0$.
2. $\lim_{x\to a} g(x) = 0$, but $\lim_{x\to a} f(x) \neq 0$, so $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.
3. $\lim_{x\to a} g(x) = 0 = \lim_{x\to a} f(x)$. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ may or may not exist; we had to do more work to decide for certain.

The $\frac{0}{0}$ form of the third limit is called an *indeterminate form*; basically, we don't know what an answer of this form means. L'Hopital's Rule, which we will learn in this section, will give us a method for handling limits that result in indeterminate forms.

Indeterminate Forms:

The following expressions, known as indeterminate forms, are things that might come up when attempting to evaluate a limit using the "plug and chug" technique:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ \infty-\infty,\ \pm\infty\cdot 0,\ 1^{\pm\infty},\ 0^0,\ \infty^0.$$

By "indeterminate form", we mean a form that doesn't make sense; we don't know what $\frac{0}{0}$ means since $\frac{0}{c} = 0$ if $c \neq 0$ and $\frac{a}{0}$ is undefined if $a \neq 0$.

Theorem 0.0.1. L'Hopital's Rule

If the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ results in one of the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the second limit exists.

The limit above can be a normal limit, a one-sided limit, or even a limit as x approaches ∞ or $-\infty$.

In other words, if we try evaluating a limit by plug and chug and end up with one of the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then we can differentiate both the numerator and denominator of the original fraction and try evaluating the limit of the new fraction.

Notice that L'Hopital's Rule has NOTHING to do with the quotient rule. We use the quotient rule to find the *derivative* of a quotient of two functions. We use L'Hopital's Rule to find the *limit* of a quotient of two functions. Note also that L'Hopital's Rule ONLY applies when the limit in question gives us one of the two specified indeterminate forms.

Examples:

Find $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Plug and chug gives us $\frac{\sin 0}{0} = \frac{0}{0}$, so we try using L'Hopital's Rule:

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{LR}{=} \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

Find $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x}$.

Again, plug and chug gives us an indeterminate form, this time $\frac{\infty}{\infty}$, so we apply L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{LR}{=} \lim_{x\to\infty} \frac{\frac{2\ln x}{x}}{1} = \frac{\infty}{\infty}$$

but we still have an indeterminate form–so we apply L'Hopital's Rule again!

$$\lim_{x\to\infty} \frac{2\ln x}{x} \overset{LR}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0.$$

Indeterminate Products:

We can not apply L'Hopital's Rule directly to an indeterminate form like $\infty \cdot 0$, but we *can* rewrite it so that L'Hopital's Rule applies. For instance, let's calculate $\lim_{x\to 0^+} \sqrt{x}\ln x$ (which results in the form $-\infty \cdot 0$):

Since $\sqrt{x}\ln x = \frac{\ln x}{\frac{1}{\sqrt{x}}}$, and $\lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}}$ results in the indeterminate form $\frac{-\infty}{\infty}$, we can now apply L'Hopital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \overset{LR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{\frac{3}{2}}}} = \lim_{x\to 0^+} \frac{-2x^{\frac{3}{2}}}{x} = \lim_{x\to 0^+} -2x^{\frac{1}{2}} = 0.$$

Indeterminate Differences:

If a limit of quotients results in the form $\infty - \infty$, adding the fractions will often enable us to use L'Hopital's Rule: in $\lim_{x\to 0^+} \cot x - \dfrac{1}{x}$, rewriting $\cot x - \frac{1}{x}$ as

$$\frac{1}{\tan x} - \frac{1}{x} = \frac{x - \tan x}{x \tan x}$$

allows us to write

$$\lim_{x\to 0^+} \cot x - \frac{1}{x} = \lim_{x\to 0^+} \frac{x - \tan x}{x\tan x};$$

since $\lim_{x\to 0^+} \dfrac{x - \tan x}{x \tan x}$ has form $\frac{0}{0}$, we can now use L'Hopital's Rule:

$$\lim_{x\to 0^+} \frac{x - \tan x}{x\tan x} \overset{LR}{=} \lim_{x\to 0^+} \frac{1 - \sec^2 x}{\tan x + x\sec^2 x}.$$

Once again, we have $\frac{0}{0}$ form, so we try L'Hopital's Rule again:

$$\lim_{x\to 0^+} \frac{1-\sec^2 x}{\tan x + x\sec^2 x} \overset{LR}{=} \lim_{x\to 0^+} \frac{-2\sec^2 x\tan x}{2\sec^2 x + 2x\sec^2 x\tan x} = \frac{0}{2} = 0.$$

Indeterminate Powers:

We can not apply L'Hopital's Rule directly to an indeterminate power such as 0^0, but we can use some mathemagic to rewrite the phrase so that L'Hopital's Rule does apply.

Find $\lim\limits_{x\to 0^+} x^x$.

Since plug and chug gives us the form 0^0, we will rewrite $x^x = e^{\ln x^x} = e^{x\ln x}$. Then $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0^+} x\ln x}$. So the problem really amounts to calculating $\lim\limits_{x\to 0^+} x\ln x$.

To find $\lim\limits_{x\to 0^+} x\ln x$, we again note that the limit is an indeterminate form, this time a product $-\infty\cdot 0$; so we rewrite $x\ln x = \frac{\ln x}{\frac{1}{x}}$, and

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}.$$

Since the limit on the right now has form $\frac{\infty}{\infty}$, L'Hopital's Rule may be applied:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{LR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

We originally wanted to find $\lim\limits_{x\to 0^+} e^{x\ln x}$; since $\lim\limits_{x\to 0^+} x\ln x = 0$, we know that

$$\lim_{x\to 0^+} e^{x\ln x} = e^0 = 1.$$

So $\lim\limits_{x\to 0^+} x^x = 1$.

Forms that are not indeterminate:

The following forms are *not* indeterminate:

$$\frac{0}{\infty},\ \frac{\infty}{0},\ \infty\cdot\infty,\ \infty+\infty,\ \infty^\infty,\ 0^\infty.$$

All of the forms above have terms working towards a common goal; for instance, a limit of the form 0^∞ is identically 0 since the base is approaching 0, and raising very small numbers to increasingly large powers results in very small (close to 0) number.