Section 10.2
The Calculus of Parametric Curves

A curve defined parametrically is a function of the parameter \( t \), but may not be a function of \( x \) or \( y \). Thus we can still ask calculus questions about the curve, such as rate of change or area, but we will have to answer those questions carefully.

Rates of Change

We will begin this section by thinking about the rate of change of a parametric curve. The parameter \( t \) controls the variables \( x \) and \( y \), indeed \( x \) and \( y \) are functions of \( t \), so it certainly makes sense to consider \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \). However, we might also be curious about \( \frac{dy}{dx} \)—that is, the rate of change of \( y \) with respect to \( x \). In particular, this would be helpful if we wished to find the equation of a tangent line to the parametric curve. Unfortunately, determining this rate of change is not as straightforward as it was when we were differentiating functions \( x \).

To solve the problem, we think of \( y \) as a function of both \( x \) and \( t \), \( y = y(x(t)) \). Then using the chain rule, we see that

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},
\]

which we also write as

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.
\]

Finally, we have

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \quad \text{if } \frac{dx}{dt} \neq 0.
\]

Example. The parametric curve shown below is defined by \( x = 4 \sin t \), \( y = t^2 + 1 \):
Find the equation (in terms of $x$ and $y$) of a tangent line to the curve at $t = \pi/3$. Notice that, at $t = \pi/3$, the coordinates of the curve are $x = 4\sin(\pi/3) = 2\sqrt{3} \approx 3$, $y = (\pi/3)^2 + 1 \approx 2$. Thus we wish to find the equation of the line graphed below:

To write the equation for the tangent line, we must know its slope, i.e. $dy/dx$. Using the equation

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt}
\]
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requires us to evaluate the derivatives of our parametric equations with respect to \( t \). Since \( y(t) = t^2 + 1 \), we see that

\[
\frac{dy}{dt} = 2t;
\]

similarly, since \( x(t) = 4 \sin t \), we have

\[
\frac{dx}{dt} = 4 \cos t.
\]

At \( t = \pi/3 \), we have

\[
\frac{dy}{dt} \bigg|_{t=\pi/3} = \frac{2\pi}{3}
\]

and

\[
\frac{dx}{dt} \bigg|_{t=\pi/3} = 4 \cdot \frac{1}{2} = 2.
\]

Thus

\[
\frac{dy}{dx} \bigg|_{t=\pi/3} = \frac{\frac{2\pi}{3}}{2} = \frac{\pi}{3}.
\]

Now the equation for a line passing through \( (x_1, y_1) \) with slope \( m \) is \( y - y_1 = m(x - x_1) \); we have already determined the slope \( m \), and we also know that the line should pass through the point

\( (x_1, y_1) = (2\sqrt{3}, (\pi/3)^2 + 1) \).

Thus the equation for the tangent line is

\[
y = \frac{(\pi/3)^2}{3} + 1 + \frac{\pi}{3} (x - 2\sqrt{3}).
\]

**Arc Length**

We can also use calculus to determine the length of a parametric curve. For instance, we might want to determine the length of the curve defined by \( x = \sin 4t \), \( y = \cos t \), \( \pi/3 \leq t \leq 2\pi/3 \):
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To think about the meaning of curve length, imagine laying a piece of string on top of the curve; once we straightened the string, we could measure it with a ruler. The length would be the length of the curve.

To determine the length of a curve mathematically, we will again employ the technique we have seen so often in calculus: we make an approximation to what we want using something we know, make the approximations better, then use calculus to go from an approximation to the exact value. Given the curve defined by \( x = f(t), \ y = g(t) \), we can make a first approximation for the length of the curve by simply joining the endpoints with a line (admittedly, this first approximation for the length of the curve is poor):

\[
L \approx L_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2};
\]

however, it will be advantageous to introduce different notation; we think of

\[
x_2 - x_1 = \Delta x \quad \text{and} \quad y_2 - y_1 = \Delta y
\]

so that

\[
L \approx L_1 = \sqrt{(\Delta x)^2 + (\Delta y)^2}.
\]

If we use, say 3 lines instead of just one, then the approximation is probably closer to the actual curve length:
Then

\[ L \approx L_1 + L_2 + L_3 = \sqrt{(\Delta x_1)^2 + (\Delta y_1)^2} + \sqrt{(\Delta x_2)^2 + (\Delta y_2)^2} + \sqrt{(\Delta x_3)^2 + (\Delta y_3)^2}. \]

Of course, we get better and better approximations by using more lines:

Our approximation of the length of the curve is now

\[ L \approx \sum_{i=1}^{k} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}. \]

We will make the approximation exact by taking a limit (which should remind you of a Reimann sum). In general,

**Theorem 5.** If the curve \( C \) is defined by the parametric equations \( x = f(t) \) and \( y = g(t) \) on \([\alpha, \beta]\), and \( f'(t) \) and \( g'(t) \) are continuous on \([\alpha, \beta]\) and \( C \) is traversed exactly once as \( t \) increases from \( t = \alpha \) to \( t = \beta \), then the length of the curve \( C \) is precisely

\[ L = \int_{\alpha}^{\beta} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt. \]

**Example.** Find the length of the curve defined by the parametric equations \( x = t^2 \) and \( y = t^3 \) from \( t = 1 \) to \( t = 2 \).

Since

\[ \frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2, \]
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the formula for arc length tells us that the length of the curve is

\[
L = \int_{1}^{2} \sqrt{(2t)^2 + (3t^2)^2} \, dt \\
= \int_{1}^{2} \sqrt{4t^2 + 9t^4} \, dt \\
= \int_{1}^{2} t\sqrt{4 + 9t^2} \, dt \\
= \frac{1}{18} \cdot \frac{2}{3} \left(\sqrt{4 + 9t^2}\right)^{3/2} \bigg|_{1}^{2} \\
= \frac{1}{27} \left(\sqrt{4 + 9t^2}\right)^{3/2} \bigg|_{1}^{2} \\
= \frac{1}{27} \left(40\frac{3}{2} - 13\frac{3}{2}\right) \\
= \frac{1}{27} \left(80\sqrt{10} - 13\sqrt{13}\right).
\]

If the curve \( f(x) \) is chosen so that \( f'(x) \) is continuous on \([a, b]\), we can consider it as a parametric curve by setting \( x = t \) and \( g(t) = f(x) \), so that \( \frac{dx}{dt} = 1 \) and \( \frac{dy}{dx} = \frac{dy}{dx} = f'(x) \); then the original formula

\[
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dt
\]

becomes

\[
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx.
\]

Alternatively, if \( x = g(y) \) is defined as a function of \( y \), then the arc length of \( x \) is given by

\[
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_{a}^{b} \sqrt{1 + (g'(y))^2} \, dy.
\]

**Example.** Find the length of the curve \( x = \frac{y^3}{6} + \frac{1}{2y} \) from \( y = 2 \) to \( y = 3 \).

Since

\[
\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2};
\]

we have

\[
\left(\frac{dx}{dy}\right)^2 = \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}.
\]
so that

\[
\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}}
\]

\[
= \sqrt{\frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}}
\]

\[
= \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2}
\]

\[
= \frac{y^2}{2} + \frac{1}{2y^2}.
\]

The length of the curve is

\[
L = \int_2^3 \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, dy
\]

\[
= \int_2^3 \frac{y^2}{2} + \frac{1}{2y^2} \, dy
\]

\[
= \frac{y^3}{6} - \frac{1}{2y}\bigg|_2^3
\]

\[
= \frac{27}{6} - \frac{1}{6} - \frac{8}{6} + \frac{1}{4}
\]

\[
= \frac{18}{6} + \frac{1}{4}
\]

\[
= 3 + \frac{1}{4}
\]

\[
= \frac{13}{4}.
\]