Section 10.1

Parametric Curves

Throughout this class, we have studied curves that are functions of $x$: that is, for each $x$, there is precisely one output $f(x)$ for the function.

However, many interesting curves are not functions of $x$. For example, the curve below is definitely not the graph of a function of $x$; it fails the vertical line test:

![Graph of a curve failing the vertical line test]

Even though this curve is not a function of $x$; however, it would be advantageous if we could think of it as a function of some other variable, so that we could use our knowledge of calculus to understand the curve.

It turns out that we can do exactly this by parameterizing the curve—that is, we will think of the coordinates of each point on the curve as functions of another “hidden” variable $t$. We write

$$x = f(t), \quad y = g(t),$$

and call these the parametric equations for the curve with parameter $t$.

The curve above has parametric equations

$$x = t^2 + t, \quad y = t^2 - 3t + 2;$$

each point on the curve has coordinates

$$(x, y) = (t^2 + t, \ t^2 - 3t + 2).$$

We find a specific point on the curve by choosing a $t$ and evaluating $x$ and $y$ at that $t$. For example, when $t = 1$, we have

$$(x, y) = (2, 0);$$

this point is marked on the curve below:
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One minor note–we can think of the function $y = g(x)$ as a very simple parameterized curve if we think of $x$ as a function of $t$ defined by $x = f(t) = t$. In other words, regular functions are not terribly different from parameterized curves.

Graphing Parametric Curves

The simplest method for drawing the graph of a curve defined parametrically is by the “t table” method: find the coordinates for several points on the curve and use these coordinates to sketch a rough graph of the curve.

**Example.** Draw a graph of the curve $C$ whose coordinates are defined parametrically by $x = t^2$ and $y = t - 3$.

To get a quick sketch, we will start by filling in the table below:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To fill in the table, we simply need to evaluate the parametric equations for $x$ and $y$ at each of the values for $t$ above. For example, if $t = 0$, then $x = 0$ and $y = -3$. Similarly, we fill in the entire
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The points indicated above are plotted below:

Connecting the points, we get the curve $C$:
The curve above is both a function of \( t \) and a function of \( y \). To see this, we can solve for \( t \) in the equation \( y = t - 3 \), yielding \( t = y + 3 \). Since \( x = t^2 \), we see that

\[
x = (y + 3)^2 = y^2 + 6y + 9.
\]

The last example leads to a helpful point: it is often possible to rewrite the formula for a parameterized curve as a “Cartesian Equation”—that is, as a function of \( x \) or \( y \). To do so, you will need to:

1. Solve for \( t \) in one of the two equations.
2. Replace \( t \) in the other equation by the solution you found above.

**Example.** Let \( C \) be the curve with parametric equations

\[
x = t^2, \quad y = \sqrt{t^4 + 1}, \quad t \geq 0.
\]

Find the Cartesian equation for \( C \).

We can easily solve for \( t \) in the equation \( x = t^2 \); we have \( t = \sqrt{x} \). Replacing \( t \) in the equation for \( y \), we have

\[
y = \sqrt{\left(\sqrt{x}\right)^4 + 1} = \sqrt{x^2 + 1}, \quad x \geq 0.
\]
A Few Important Parameterizations

The two most common parameterizations that we will see are those for a circle and for a line segment. A circle of radius $r$ centered at the point $(h,k)$ has parameterization

$$x = r \cos \theta + h, \quad y = r \sin \theta + k, \quad 0 \leq \theta \leq 2\pi.$$  

More generally, an ellipse centered at $(h,k)$ has parameterization

$$x = r_1 \cos \theta + h, \quad y = r_2 \sin \theta + k, \quad 0 \leq \theta \leq 2\pi.$$  

A line segment joining the points $(x_1,y_1)$ and $(x_2,y_2)$ may be parameterized as follows:

$$x = (1 - t)x_1 + tx_2, \quad y = (1 - t)y_1 + ty_2, \quad 0 \leq t \leq 1.$$  

Example. Find parametric equations for the ellipse centered at the origin passing through the points $(3,0)$ and $(0,2)$.

Since the ellipse is centered at the origin, the parametric equations for our curve are

$$x = r_1 \cos \theta, \quad y = r_2 \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$  

We need to find the values for $r_1$ and $r_2$, which are essentially the “radii”. To find them, note that the ellipse should pass through the points $(3,0)$ and $(-3,0)$. Thus we need to scale up the $x$ coordinate by 3; so $r_1 = 3$, and similarly $r_2 = 2$. Thus the the parametric equations for the curve are

$$x = 3 \cos \theta, \quad y = 2 \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$  

The curve is graphed below: