# An Egocentric Logic of De Dicto and De Re Knowing Who

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**Abstract**

The article proposes de dicto and de re versions of "knowing-who" modalities as well as studies the interplay between them and modalities "knows" and "for all agents". It shows that neither of these four modalities is definable through a combination of the three others. In addition, a sound and complete logical system describing the properties of de dicto "knows who", "knows", and "for all agents" modalities is presented.

## 1 Introduction

The distinction between de re and de dicto has long been a topic of study in the philosophy of language [23, 16, 3, 15, 14, 26, 6]. Outside philosophy, this distinction has also been discussed in settings such as disability rights and genocide [1]. There are multiple views on how exactly this distinction can be defined [21], but the general idea is that "de re" refers to an object and "de dicto" to the name of the object.

De re/de dicto distinction manifests itself in many different settings: awareness, knowledge, preferences, and others. In this article, we study de re and de dicto versions of *knowing who.* As an example, consider a double blind date setting where two gentlemen, Bob and Doug, have invited two ladies on a dinner date. Each lady knows that the names of the gentlemen are Bob and Doug, but neither of them knows what the gentlemen look like. To distinguish the gentlemen, we assume that one of them has dark eyes and a light bow tie, while the other has light eyes and a dark bow tie. Similarly, one of the ladies has dark eyes and a light hair tie, while the other has light eyes and a dark hair tie, see Figure 1.

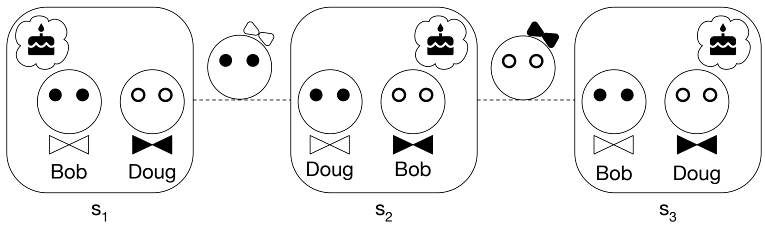


Figure 1: Double blind date example.

By the end of the dinner, the ladies learn that one of the gentlemen is celebrating his birthday today. Let us also assume that by the end of the dinner just enough information is revealed so that the dark-eyed lady allows for only two possibilities: either (i) the dark-eyed gentleman is Bob and today is his birthday or (ii) the dark-eyed gentleman is Doug and today is not his birthday. We depict these two possibilities as epistemic states $s_1$ and $s_2$ in Figure 1. These states are indistinguishable to the dark-eyed lady. At the same time, the light-eyed lady came to the conclusion that the only two possible cases are those represented by states $s_2$ and $s_3$ in the same diagram. Of course, a thoughtful reader would see by now that the actual epistemic state is $s_2$, but neither of the two ladies knows this. Note that both ladies know whose birthday is today, but they know this in different senses: the dark-eyed lady knows that Bob has a birthday and the light-eyed lady knows that the light-eyed gentleman has a birthday. The first of them knows *the name* of the birthday gentleman, while the other knows who he is as an *object* of physical appearance. We say that first of them knows *de dicto* whose birthday it is, while the second knows this *de re*[1].

## 2 Formalizations

The most straightforward way to formalize the introductory example is to use epistemic logic with quantifiers over agents [19]:

$$\mathsf{K}_{\text{dark-eyed lady}} \forall x \in Agents(HasName(x, Bob) \to HasBirthday(x))),$$
$$\forall x \in Agents\,(HasName(x, Bob) \to \mathsf{K}_{\text{light-eyed lady}}(HasBirthday(x))).$$

Wang and Seligman proposed a logical system for capturing properties of non-rigid names [26]. Their system includes an update modality $[x := t]$ that

[1]Note that the terms de re and de dicto are also sometimes used outside of the context of the distinction between an object and the name of the object [10, 2, 9].

assigns to a variable $x$ the current value of the name term $t$. Our examples can be captured in their language as follows:

$$\mathsf{K}_{\text{dark-eyed lady}}(HasBirthday(Bob)),$$
$$[x := Bob]\mathsf{K}_{\text{light-eyed lady}}(HasBirthday(x)).$$

However, neither of the above formalizations completely captures the "knowing who" in the current article. To do this, one would need to further extend the above systems with quantifiers over names. For example, in the case of the logic with quantifiers,

$$\exists n \in Names\,\mathsf{K}_{\text{dark-eyed lady}}\forall x \in Agents(HasName(x,n) \to HasBirthday(x))),$$
$$\exists n \in Names\,\forall x \in Agents\,(HasName(x,n) \to \mathsf{K}_{\text{light-eyed lady}}(HasBirthday(x))).$$

In this article, we propose a much simpler and more elegant language for capturing the notions of de re and de dicto knowing who which does not involve quantifiers or even variables. We achieve this by using an *egocentric* setting.

# 3 Egocentric Logics

The semantics of traditional modal logics is defined in terms of a binary satisfaction relation $s \Vdash \varphi$ between a state and formula. In such a setting, the formula $\varphi$ captures a property of state $s$. For example, the statement $s \Vdash$ "the blind date is at 6 pm" means that state $s$ has the property of the blind date taking place at 6 pm. Prior proposed to consider egocentric logics in which statements capture properties of agents rather than worlds [22]. Using his approach, for example, we can write $a \Vdash$ "has a birthday today" to denote the fact that agent $a$'s birthday is today. One can use Boolean connectives to express more complicated statements in an egocentric setting. For example, the statement

$$a \Vdash \text{"has a birthday today"} \wedge \text{"has to work today"}.$$

means that although agent $a$'s birthday is today, the agent still has to work. Seligman, Liu, and Girard proposed friendship modality $\mathsf{F}$ in the egocentric setting [24, 25]. Using this modality one can express the fact that agent $a$'s birthday is today, but all his friends have to work:

$$a \Vdash \text{"has a birthday today"} \wedge \mathsf{F}(\text{"has to work today"}).$$

Modality $\mathsf{F}$ is also used in [4, 5]. Jiang and Naumov introduced "likes those who" modality $\mathsf{L}$. It can be used to state that agent $a$ likes those who smile: $a \Vdash \mathsf{L}(\text{"smiles"})$. Egocentric modalities can be nested in the usual way. For example, the statement $a \Vdash \mathsf{FL}(\text{"smiles"})$ means that all friends of agent $a$ like those who smile and the statement $a \Vdash \neg\mathsf{L}\neg\mathsf{L}(\text{"smiles"})$ means that agent $a$ does not like those who do not like people who smile.

Grove and Halpern considered a version of the egocentric setting in which statements capture the properties of agents in a given world. To do this, they used ternary satisfaction relation. For example, in their setting, sentence

$$(a, s) \Vdash \text{“has birthday today”}$$

means that agent $a$ has birthday *in state* $s$. In such a setting, they used indistinguishability relation between states to define “knows about itself” modality $\mathsf{K}$. Using this modality, one can express that in state $s$ agent $a$ knows that agent $a$ has a birthday today:

$$(a, s) \Vdash \mathsf{K}\text{“has birthday today”}.$$

Modality $\mathsf{K}$ can also be combined with other modalities. For example, the statement

$$(a, s) \Vdash \mathsf{KF}(\text{“has to work today”})$$

means that agent $a$ knows in state $s$ that all its friends have to work today. In [20], two of us presented “knows how to tell apart” modality that we denote here by $\mathsf{T}$. Using this modality, one can write that in state $s$ agent $a$ knows how to tell apart those who smile in state $s$ from those who do not: $(a, s) \Vdash \mathsf{T}(\text{“smiles”})$. In Section 7, we further discuss the difference between modality $\mathsf{T}$ and the two “knowing who” modalities from the current article.

In this article, we use an egocentric setting to define and study de re and de dicto versions of knowing-who modalities.

## 4 Outline

This article is organized as follows. First, we define the class of egocentric models used to give the formal semantics of our knowing-who modalities. This class of models has been originally proposed by Grove and Halpern. Then, we illustrate this semantics in another example and give formal syntax and semantics of our logical system. The system, in addition to de dicto and de re modalities, also contains modalities “knows” and “for all agents”.

The technical results are divided into two groups. First, we show that neither of the four modalities is definable through a combination of the three others. This gives a justification for developing a formal logical system that would include all four modalities. As a step towards this goal, in the second part of the paper we give a sound and complete logical system describing the interplay between “de dicto knows who”, “knows”, and “for all agents” modalities. From our experience, a complete axiomatization of just “de re knows who”, “knows”, and “for all agents” (without “de dicto knows who”) would require new techniques beyond what is developed in the current work. We leave this problem, as well as a joint axiomatization of all four modalities, for future research.

The preliminary version of this work with only an informal sketch of the proof of completeness and without the undefinability results has appeared as [7].

## 5 Grove-Halpern Models

The formal semantics of names that we use in this paper was first proposed by Grove and Halpern to study modality "for all agents with a given name" [12].

**Definition 1** *A tuple $(S, A, P, \{\sim_a\}_{a\in A}, N, I, \pi)$ is called a model if*

1. *$S$ is an arbitrary set of "states",*
2. *$A$ is an arbitrary set of "agents",*
3. *$P$ is a function that maps each agent $a \in A$ into a set of states $P(a) \subseteq S$ in which the agent is "present",*
4. *$\sim_a$ is an "indistinguishability" equivalence relation on set $P(a)$ for each agent $a \in A$,*
5. *$N$ is a nonempty set of "names",*
6. *$I \subseteq A \times S \times N \times A$ is an "identification mechanism" relation satisfying the following two conditions:*
   (a) *for each $a \in A$, each $s \in P(a)$, and each $n \in N$, there is at least one agent $a' \in A$ such that $(a, s, n, a') \in I$,*
   (b) *for each $a \in A$, each $s \in P(a)$, each $n \in N$, and each agent $a' \in A$, if $(a, s, n, a') \in I$, then $s \in P(a')$,*
7. *for each propositional variable $p$, set $\pi(p)$ is an arbitrary set of pairs $(a, s)$ such that $a \in A$ and $s \in P(a)$.*

Figure 1 depicts a Grove-Halpern model for the double-blind date example. Grove-Halpern models use states and indistinguishability relation $\sim_a$ to capture knowledge in almost the same way as it is done in Kripke models for the epistemic logic S5. The diagram in Figure 1 depicts three states, $s_1$, $s_2$, and $s_3$. There are four agents in this example: two gentlemen and two ladies. To avoid overcrowding the picture in Figure 1, we have chosen not to draw the pictures of the ladies inside the states. However, they are assumed to be present in all three states. Thus, $P(a) = \{s_1, s_2, s_3\}$ for each agent $a$. In general, Grove-Halpern models allow some of the agents not to be present in some of the states. This could be used, for example, to model the situation when one of the agents is not aware of the existence of some other agents.

The indistinguishability relation of an agent is an equivalence relation on the states in which the agent is present. In our example, states $s_1$ and $s_2$ are indistinguishable by the dark-eyed lady and states $s_2$ and $s_3$ are indistinguishable by the light-eyed lady. We assume that the gentlemen can distinguish all three states.

In our example, the set of names $N$ consists of Bob and Doug. The identification mechanism $I$ is the key part of defining a Grove-Halpern model. This mechanism specifies which name could be used to refer to which agent. Grove-Halpern models take one of the most general approaches to assigning names to agents. They allow names like "my mother" that might refer to different women when used by different people. Thus, the meaning of a name is assumed to be *agent-specific*. They also allow names like "my best friend" that might refer to different people in different states. Thus, the meaning of a name is assumed to be *state-specific*. Furthermore, it is assumed that an agent might use different names to refer to the same person. Finally, the models allow names like "locksmith" that the same person in the same state might use to refer to two different people. If one says that she knows who, locksmith, can open the lock, then we interpret this as her saying that she knows that any locksmith can do it. To support all these features, an identification mechanism $I$ is specified as a set of tuples $(a, s, n, a') \in A \times S \times N \times A$. If $(a, s, n, a') \in I$, then agent $a$ in state $s$ might use name $n$ to refer to agent $a'$. In our example, the names are state-specific, but not agent-specific. They also uniquely identify an agent in each state.

Despite allowing very general identification mechanisms, we impose on them two restrictions captured by conditions 6(a) and 6(b) of Definition 1. The first of these two conditions states that for any agent $a$, any state $s \in P(a)$, and any name $n \in N$, there must exist at least one agent $a'$ that agent $a$ refers to by name $n$ in state $s$. The second condition requires that any of the above agents $a'$ must be present in state $s$. In other words, we want to exclude cases when the dark-eyed lady would claim that she knows who, say the Santa Claus, has the birthday, when there is no single person who is Santa Claus either at all or in the current state. We introduce these two conditions on namespaces because we believe that without them our formal definition of "knowing-who" modality, see Definition 2, does not reflect the informal meaning of "knowing who".

Another important difference between the Grove-Halpern models and the standard Kripke semantics for epistemic logic S5 is that valuation function $\pi$ maps propositional variables not into sets of states, but into sets of pairs $(a, s)$ consisting of an agent $a$ and a state $s \in P(a)$ in which agent $a$ is present. In other words, propositional variables are interpreted as sentences in which the subject is omitted. Grove and Halpern call them *relative* sentences. Informally, set $\pi(p)$ is the set of all pairs $(a, s)$ such that statement $p$ is true about agent $a$ in state $s$. We further discuss this in the next section.

Grove and Halpern first introduced Definition 1 without conditions 6(a) and 6(b)[12]. Later they added condition 6(b) [13]. In his third work, Grove again added condition 6(a), but in a form stronger than ours: "exactly one" agent [11] instead of "at least one".

# 6 Mafia Game Example

In this section, we give another example of a Grove-Halpern model and simultaneously introduce the syntax of *de re* and *de dicto knowing-who* modalities.

The example is based on the Mafia game. In this game, several players are

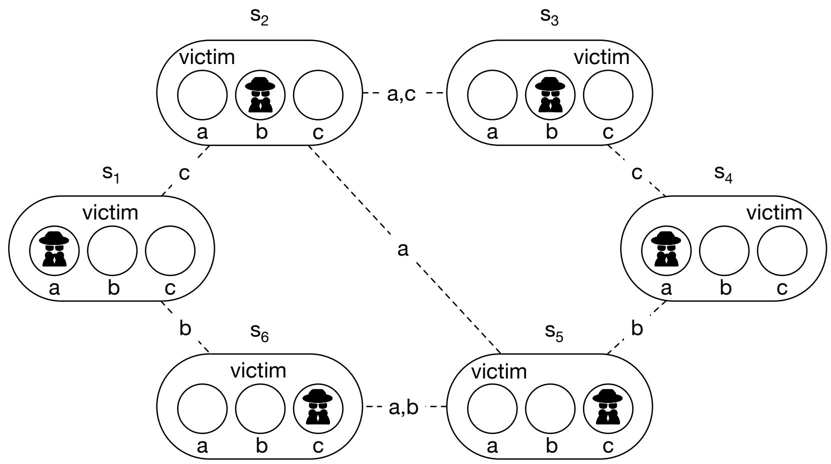


Figure 2: Mafia game example.

selected (in secret) to be the "mafia". The game consists of multiple rounds. At the beginning of each round, the host of the game first asks all players to close their eyes and, next, the members of the mafia to open their eyes. In silence, using hand gestures only, the members of the mafia select one of the non-mafia players to be "killed". The host then first asks the mafia to close their eyes and, next, instructs everyone to open their eyes. After this, the host announces who is killed by the mafia. This player leaves the game. Next, all remaining players of the game discuss who the mafia could be, select one of the players and "kill" that player in the hope that it is one of the members of the mafia. The host announces if the killed player belonged to the mafia or not. This player leaves the game, after which the next round of the game starts. The game ends when either all members of the mafia or all the non-mafia players are killed. If only one member of the mafia and one non-mafia member are left, the host announces a draw.

In our example, we assume that there are only three players: agent $a$, agent $b$, and agent $c$. We consider the moment of the game after the mafia has chosen the "victim" to be killed next, but the host has not announced this information yet. The six possible states of the game are depicted in Figure 2. In this diagram, the member of the mafia is depicted in the hat and the "victim" agent is labelled accordingly. The indistinguishability relation captures the information known to each agent at that moment. For example, states $s_1$ and $s_4$ are *distinguishable* by agent $a$ because agent $a$ is the mafia and this agent has chosen different victims in these two states. The same two states are *indistinguishable* by agents $b$ and $c$ because these two agents do not know yet who is the victim. To keep the diagram readable, we assume that the indistinguishability relation is the reflexive and transitive closure of the relation shown using dashed lines.

Note that $a$, $b$, and $c$ are the actual agents, not their names. We assume that there are only two names: "mafia" and "non-mafia". In this example, the names are state-specific: agent $a$ is referred to as "mafia" in state $s_1$ but not in state $s_2$. At the same time, in this example, the names are *not* agents-specific: each of the three agents can refer to agent $a$ in state $s_1$ as "mafia". Finally, we will treat "is victim" as the only propositional variable in this example. The value of the function $\pi$ for this variable includes, for instance, the pair $(b, s_1)$ because agent $b$ is the victim in state $s_1$. To state that agent $b$ is the victim in state $s_1$, we write

$$(b, s_1) \Vdash \text{"is victim"}.$$

In general, we write $(g, s) \Vdash \varphi$ if in state $s$ statement $\varphi$ is true *about* agent $g$. The idea to consider satisfaction as a ternary relation between an agent, a state, and a formula goes back to the original works by Grove and Halpern [12, 13, 11]. Note that the subject of statement $\varphi$ is agent $g$ and it is not, generally speaking, mentioned in the statement $\varphi$ itself.

Although agent $b$ is the victim in state $s_1$, the agent does not know this *about itself* because agent $b$ cannot distinguish state $s_1$ from state $s_5$ where the agent is not the victim. We write this as

$$(b, s_1) \Vdash \neg\mathsf{K}(\text{"is victim"}).$$

In general, we write $(g, s) \Vdash \mathsf{K}\varphi$ if in state $s$ agent $g$ knows that statement $\varphi$ *about itself* is true.

Next, note that because agent $a$ is the mafia in state $s_1$, this agent knows who, agent $b$, is the victim. Because $b$ is the actual agent, not a name, agent $a$ knows *de re* who is the victim. We write this as

$$(a, s_1) \Vdash \mathsf{R}(\text{"is victim"}).$$

In general, we write $(g, s) \Vdash \mathsf{R}\varphi$ if in state $s$ agent $g$ knows de re an agent about whom statement $\varphi$ is true. Note that agent $b$ cannot distinguish state $s_1$ from state $s_5$, where the victim is agent $a$. Thus,

$$(b, s_1) \Vdash \neg\mathsf{R}(\text{"is victim"}).$$

However, agent $b$ knows that the mafia cannot kill one of its members. So agent $b$ de dicto knows who, the mafia, is not the victim:

$$(b, s_1) \Vdash \mathsf{D}\neg(\text{"is victim"}).$$

Agent $b$ also knows that the mafia knows de re the agent who is the victim. Thus, agent $b$ knows de dicto who knows de re who is the victim:

$$(b, s_1) \Vdash \mathsf{DR}(\text{"is victim"}).$$

At the same time, agent $b$ does not know de re who knows de re who is the victim:

$$(b, s_1) \Vdash \neg\mathsf{RR}(\text{"is victim"}).$$

This is because only the mafia knows de re who is the victim and agent $b$ does not know de re who is the mafia.

In this article, in addition to modalities $\mathsf{K}$, $\mathsf{R}$, and $\mathsf{D}$, we also consider modality $\mathsf{A}$ which means "for all agents". For example,

$$(b, s_1) \nVdash \mathsf{A}(\text{“is victim”})$$

because not all agents are the victims. The dual modality $\neg\mathsf{A}\neg$ means "there is an agent". For example,

$$(b, s_1) \Vdash \neg\mathsf{A}\neg(\text{“is victim”})$$

because there is at least one victim in state $s_1$. We have observed above that agent $b$ does not know de re who the victim is. It could also be shown that agent $b$ does not know this de dicto. However, agent $b$ knows that there is a victim:

$$(b, s_1) \Vdash \mathsf{K}\neg\mathsf{A}\neg(\text{“is victim”}).$$

# 7 Syntax and Semantics

In this section, we define the formal syntax and semantics of our logical system. The language $\Phi$ of the system is defined by the grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \to \psi \mid \mathsf{A}\varphi \mid \mathsf{K}\varphi \mid \mathsf{D}\varphi \mid \mathsf{R}\varphi.$$

We read $\mathsf{A}\varphi$ as "$\varphi$ is true for all agents", $\mathsf{K}\varphi$ as "knows $\varphi$ about itself", $\mathsf{D}\varphi$ as "knows de dicto somebody for whom $\varphi$ is true", and $\mathsf{R}\varphi$ as "knows de re somebody for whom $\varphi$ is true".

**Definition 2** *For any model $(S, A, P, \{\sim_a\}_{a\in A}, N, I, \pi)$, any agent $a \in A$, any state $s \in P(a)$, and any formula $\varphi \in \Phi$, satisfiability relation $(a, s) \Vdash \varphi$ defined as follows:*

1. *$(a, s) \Vdash p$ if $(a, s) \in \pi(p)$,*
2. *$(a, s) \Vdash \neg\varphi$ if $(a, s) \nVdash \varphi$,*
3. *$(a, s) \Vdash \varphi \to \psi$ if $(a, s) \nVdash \varphi$ or $(a, s) \Vdash \psi$,*
4. *$(a, s) \Vdash \mathsf{A}\varphi$ if $(a', s) \Vdash \varphi$ for each agent $a' \in A$ such that $s \in P(a')$,*
5. *$(a, s) \Vdash \mathsf{K}\varphi$ if $(a, s') \Vdash \varphi$ for each state $s' \in P(a)$ such that $s \sim_a s'$,*
6. *$(a, s) \Vdash \mathsf{D}\varphi$ when there is a name $n \in N$ such that for each state $s' \in P(a)$ and each agent $a' \in A$, if $s \sim_a s'$ and $(a, s', n, a') \in I$, then $(a', s') \Vdash \varphi$.*
7. *$(a, s) \Vdash \mathsf{R}\varphi$ when there is an agent $a' \in A$ such that for each state $s' \in P(a)$ if $s \sim_a s'$, then $s' \in P(a')$ and $(a', s') \Vdash \varphi$.*

Note that item 5 of the above definition only considers states $s'$ in which agent $a$ is present. In essence, this captures the assumption that the agent is self-aware (knows that it exists) in state $s$. The same applies to items 6 and 7. In addition, the quantifier over agent $a'$ in item 6 is universal. It means that if there are multiple agents with the same name, then we require each of them to have property $\varphi$. For example, if agent $a$ de dicto knows who, a locksmith, can open the door, then it means that any locksmith should be able to do this.

The knowing how to tell them apart modality $\mathsf{T}$ from [20] is defined as:

$(a, s) \Vdash \mathsf{T}\varphi$ *when for each agent* $a' \in A$ *and any states* $t, t' \in S$*, if* $s \sim_a t$*,* $s \sim_a t'$*, and* $(a', t) \Vdash \varphi$*, then* $(a', t') \Vdash \varphi$*.*

Note that while de re knowing who modality $\mathsf{R}\varphi$ expresses *knowledge of a specific agent* $a'$ for whom $\varphi$ is true, the knowing how to tell them apart modality $\mathsf{T}$ expresses *an ability* of agent $a$ to decide *for each* agent $a'$ if $a'$ has property $\varphi$.

Neither of the modalities $\mathsf{T}$ and $\mathsf{R}$ implies the other. For example, any agent can tell apart somebody who is Santa Claus (since there is no single Santa Claus), but an agent cannot know who is Santa Claus (again because there is no such thing). On the other hand, even if an agent knows who can solve a math puzzle (the math teacher), the agent might not be able to tell apart all those who can solve the puzzle.

# 8 Undefinability results

In this section, we show that modalities $\mathsf{A}$, $\mathsf{K}$, $\mathsf{D}$, and $\mathsf{R}$ are not definable through each other. We start with the undefinability of the modality $\mathsf{R}$ because the proof, in this case, is, perhaps, the most interesting.

## 8.1 Undefinability of $\mathsf{R}$

In this subsection, we show that modality $\mathsf{R}$ cannot be defined through modalities $\mathsf{A}$, $\mathsf{K}$, and $\mathsf{D}$. To prove this, we construct two Grove-Halpern models which are indistinguishable in the language without modality $\mathsf{R}$, that we denote by $\Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$, and are distinguishable using modality $\mathsf{R}$.

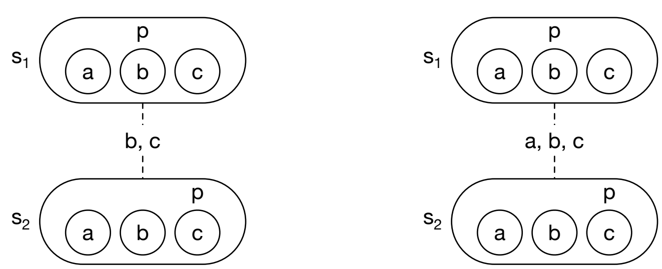


Figure 3: Two Grove-Halpern models.

The two models are depicted in Figure 3. We refer to these two models as the left and the right models. Both of these models have only two states: $s_1$ and $s_2$ and only three agents: $a$, $b$, and $c$. All agents are present in all states. In the left model, the two states are distinguishable to agent $a$ and indistinguishable to agents $b$ and $c$. In the right model, neither of the agents can distinguish the two states.

We assume that the set $N$ of names in both models consists of just a single name $se$ (self). In each world of each of the two models, each agent can use this name to refer to itself. Thus, $I = \{(x, y, se, x) \mid x \in \{a, b, c\}, y \in \{s_1, s_2\}\}$. Note that, by Definition 2, for such a mechanism, agents $a$ and $a'$ in item 6 of Definition 2 must be the same. As a result, modalities $\mathsf{D}$ and $\mathsf{K}$ for such mechanisms are equivalent:

$$(x, y) \Vdash \mathsf{D}\varphi \quad \text{iff} \quad (x, y) \Vdash \mathsf{K}\varphi, \tag{1}$$

for any agent $x \in \{a, b, c\}$, any world $y \in \{s_1, s_2\}$, and any formula $\varphi \in \Phi$.

Without loss of generality, we assume that the language contains only a single propositional variable $p$. Finally, for both models, let $\pi(p)$ be the set $\{(b, s_1), (c, s_2)\}$, see Figure 3.

By $\Vdash_l$ and $\Vdash_r$ we denote the satisfaction relation of the left and the right models, respectively. Informally, the next lemma follows from the symmetry of the right model. The formal proof could be done by induction on the structural complexity of formula $\psi$.

**Lemma 1** *For any formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$,*

1. $(a, s_1) \Vdash_r \varphi$ *iff* $(a, s_2) \Vdash_r \varphi$,
2. $(b, s_1) \Vdash_r \varphi$ *iff* $(c, s_2) \Vdash_r \varphi$,
3. $(c, s_1) \Vdash_r \varphi$ *iff* $(b, s_2) \Vdash_r \varphi$.

Proof. We prove all three statements of the lemma *concurrently* by induction on the structural complexity of formula $\varphi$. If formula $\varphi$ is propositional variable $p$, then the first statement of the lemma is true by item 1 of Definition 2, because $(a, s_1) \notin \pi(p)$ and $(a, s_2) \notin \pi(p)$ in the right model. Similarly, the second statement is true because $(b, s_1) \in \pi(p)$ and $(c, s_2) \in \pi(p)$. Finally, the third statement is true because $(c, s_1) \notin \pi(p)$ and $(b, s_2) \notin \pi(p)$.

The case when formula $\varphi$ is a negation or an implication follows from item 2 and item 3 of Definition 2 and the induction hypothesis in the standard way.

Suppose formula $\varphi$ has the form $\mathsf{A}\psi$. It suffices to show that $(x, s_1) \Vdash_r \mathsf{A}\psi$ iff $(y, s_1) \Vdash_r \mathsf{A}\psi$ for any agents $x, y \in \{a, b, c\}$. Indeed, note that $(x, s_1) \Vdash_r \mathsf{A}\psi$ iff $\forall z \in \{a, b, c\}((z, s_1) \Vdash \psi)$ by item 4 of Definition 2. Also, $\forall z \in \{a, b, c\}((z, s_1) \Vdash \psi)$ iff $\forall z \in \{a, b, c\}((x, s_2) \Vdash \psi)$ by the induction hypothesis. Finally, note that $\forall z \in \{a, b, c\}((z, s_2) \Vdash \psi)$ iff $(y, s_2) \Vdash_r \mathsf{A}\psi$ by item 4 of Definition 2.

Let formula $\varphi$ have the form $\mathsf{K}\psi$. We consider the three statements of the lemma separately.

1. Note that the statements $(a, s_1) \Vdash_r \mathsf{K}\psi$ and $(a, s_2) \Vdash_r \mathsf{K}\psi$ are both equivalent to the statement $\forall s' \in \{s_1, s_2\}((a, s') \Vdash_r \psi)$ by item 5 of Definition 2 because agent $a$ cannot distinguish states $s_1$ and $s_2$ in the right model.

2. Observe that $(b, s_1) \Vdash_r \mathsf{K}\psi$ iff $\forall s' \in \{s_1, s_2\}((b, s') \Vdash_r \psi)$ by item 5 of Definition 2 because agent $b$ cannot distinguish states $s_1$ and $s_2$ in the right model. Also, $\forall s' \in \{s_1, s_2\}((b, s') \Vdash_r \psi)$ iff $\forall s' \in \{s_1, s_2\}((c, s') \Vdash_r \psi)$ by the induction hypothesis. Finally, $\forall s' \in \{s_1, s_2\}((c, s') \Vdash_r \psi)$ iff $(c, s_2) \Vdash_r \mathsf{K}\psi$ by item 5 of Definition 2 because agent $c$ cannot distinguish states $s_1$ and $s_2$ in the right model.

   The proof of the third statement is similar.

If formula $\varphi$ has the form $\mathsf{D}\psi$, then the statement of the lemma follows from the previous case and statement (1). ⊠

The next lemma is the key step in the proof of the undefinability. It shows that the left and the right models are indistinguishable in language $\Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.

**Lemma 2** *$(x, s) \Vdash_l \varphi$ iff $(x, s) \Vdash_r \varphi$, for any agent $x \in \{a, b, c\}$, any state $s \in \{s_1, s_2\}$, and any formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.*

Proof. We prove the lemma by induction on the structural complexity of formula $\varphi$. Recall that $\pi(p) = \{(b, s_1), (c, s_2)\}$ for both models. Thus, $(x, s) \Vdash_l p$ iff $(x, s) \Vdash_r p$ for any agent $x \in \{a, b, c\}$, any state $s \in \{s_1, s_2\}$ by item 1 of Definition 2. The case when formula $\varphi$ is a negation or an implication follows from item 2 and item 3 of Definition 2 in the standard way.

Suppose that formula $\varphi$ has the form $\mathsf{A}\psi$. Note that statement $(x, s) \Vdash_l \mathsf{A}\psi$ is equivalent to the statement $\forall y \in \{a, b, c\}((y, s) \Vdash_l \psi)$ by item 4 of Definition 2. The last statement is equivalent to $\forall y \in \{a, b, c\}((y, s) \Vdash_r \psi)$ by the induction hypothesis. Finally, statement $\forall y \in \{a, b, c\}((y, s) \Vdash_r \psi)$ is equivalent to the statement $(x, s) \Vdash_r \mathsf{A}\psi$ again by item 4 of Definition 2.

Assume that formula $\varphi$ has the form $\mathsf{K}\psi$. We consider the following three cases separately:
*Case I*: $x = a$. Note that agent $a$ can distinguish all states in the left model. Thus, $(a, s) \Vdash_l \mathsf{K}\psi$ iff $(a, s) \Vdash_l \psi$ by item 5 of Definition 2. Also, $(a, s) \Vdash_l \psi$ iff $(a, s) \Vdash_r \psi$ by the induction hypothesis. Next, observe that $(a, s) \Vdash_r \psi$ iff $\forall s' \in \{s_1, s_2\}((a, s') \Vdash_r \psi)$ by item 1 of Lemma 1. Finally, $\forall s' \in \{s_1, s_2\}((a, s') \Vdash_r \psi)$ iff $(a, s) \Vdash_r \mathsf{K}\psi$ by item 5 of Definition 2 because agent $a$ cannot distinguish the states of the right model.
*Case II*: $x \neq a$. Then, agent $x$ cannot distinguish the states of the left model. Thus, $(x, s) \Vdash_l \mathsf{K}\psi$ iff $\forall s' \in \{s_1, s_2\}((x, s') \Vdash_l \psi)$ by item 5 of Definition 2. Note that $\forall s' \in \{s_1, s_2\}((x, s') \Vdash_l \psi)$ iff $\forall s' \in \{s_1, s_2\}((x, s') \Vdash_r \psi)$ by the induction hypothesis. Finally, $\forall s' \in \{s_1, s_2\}((x, s') \Vdash_r \psi)$ iff $(x, s) \Vdash_r \mathsf{K}\psi$ by item 5 of Definition 2 because agent $x$ cannot distinguish the states of the right model.

If formula $\varphi$ has the form $\mathsf{D}\psi$, then the statement of the lemma follows from the previous case and statement (1). ⊠

The next lemma follows from item 7 of Definition 2 and the definition of the left and the right models, see Figure 3.

**Lemma 3** *$(a, s_1) \Vdash_l \mathsf{R}p$ and $(a, s_1) \nVdash_r \mathsf{R}p$.*

Finally, the theorem below follows from the two previous lemmas.

**Theorem 1** *Modality $\mathsf{R}$ is not definable in language $\Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.*

## 8.2 Undefinability of $\mathsf{K}$

In this subsection, we prove that modality $\mathsf{K}$ is not definable through modalities $\mathsf{A}$, $\mathsf{D}$, and $\mathsf{R}$. Without loss of generality, we again assume that our language has a single propositional variable $p$. To prove the undefinability, let us consider a single Grove-Halpern model depicted in Figure 4. We assume that set $N$ contains a single name $Ann$. In each state, each agent uses this name to refer to agent $a$. That is, $I = \{(x, y, Ann, a) \mid x \in \{a, b\}, y \in \{s_1, s_2\}\}$.

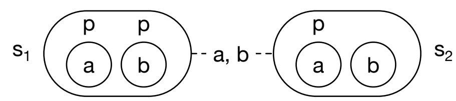


Figure 4: Grove-Halpern model.

The next lemma shows that agents $a$ and $b$ of this model are not distinguishable in state $s_1$ using language $\Phi^{\mathsf{A},\mathsf{D},\mathsf{R}}$. Later, we show that these agents are distinguishable in the same state $s_1$ using modality $\mathsf{K}$.

**Lemma 4** *$(a, s_1) \Vdash \varphi$ iff $(b, s_1) \Vdash \varphi$, any formula $\varphi \in \Phi^{\mathsf{A},\mathsf{D},\mathsf{R}}$.*

PROOF. We prove the lemma by induction on the structural complexity of formula $\varphi$. Note that $(a, s_1) \Vdash p$ and $(b, s_1) \Vdash p$. Thus, the statement of the lemma holds for propositional variable $p$. If formula $\varphi$ is a negation or an implication, then the statement of the lemma follows from the induction hypothesis in the standard way.

If formula $\varphi$ has the form $\mathsf{D}\psi$, then, by item 6 of Definition 2 and because the model has a single name $Ann$, the statements $(a, s_1) \Vdash \varphi$ and $(b, s_1) \Vdash \varphi$ are both equivalent to the conjunction of the statements $(a, s_1) \Vdash \psi$ and $(a, s_2) \Vdash \psi$.

Suppose that formula $\varphi$ has the form $\mathsf{R}\psi$. Note that, because states $s_1$ and $s_2$ are indistinguishable by agent $a$ and agent $b$, statements $(a, s_1) \Vdash \mathsf{R}\psi$ and $(b, s_1) \Vdash \mathsf{R}\psi$ are both equivalent to the statement $\exists c \forall s\, ((c, s) \Vdash \psi)$ by item 7 of Definition 2.

Finally, let $\varphi$ have the form $\mathsf{A}\psi$. Then, the statement $(a, s_1) \Vdash \mathsf{A}\psi$ and the statement $(b, s_1) \Vdash \mathsf{A}\psi$ are both equivalent to the statement $\forall c\, ((c, s_1) \Vdash \psi)$ by item 4 of Definition 2. $\boxtimes$

The next lemma shows that agents $a$ and $b$ are distinguishable in state $s_1$ using modality $\mathsf{K}$. The lemma follows from item 5 of Definition 2.

**Lemma 5** *$(a, s_1) \Vdash \mathsf{K}p$ and $(b, s_1) \nVdash \mathsf{K}p$.*

Combined, the two previous lemmas imply the desired result.

**Theorem 2** *Modality $\mathsf{K}$ is not definable in language $\Phi^{\mathsf{A},\mathsf{D},\mathsf{R}}$.*

## 8.3 Undefinability of $\mathsf{A}$

In this subsection, we prove that modality $\mathsf{A}$ is not definable through modalities $\mathsf{K}$, $\mathsf{D}$, and $\mathsf{R}$. Without loss of generality, we again assume that our language has a single propositional variable $p$. To prove the undefinability, we use the same Grove-Halpern model depicted in Figure 4 as in the previous section.

**Lemma 6** *$(a, s_1) \Vdash \varphi$ iff $(a, s_2) \Vdash \varphi$, where $\varphi \in \Phi^{\mathsf{K},\mathsf{D},\mathsf{R}}$.*

Proof. We prove the lemma by induction on the structural complexity of formula $\varphi$. Note that $(a, s_1) \Vdash p$ and $(a, s_2) \Vdash p$. Thus, the statement of the lemma holds for propositional variable $p$. If formula $\varphi$ is a negation or an implication, then the statement of the lemma follows from the induction hypothesis in the standard way.

Suppose formula $\varphi$ has the form $\mathsf{K}\psi$. Note that statement $(a, s_1) \Vdash \mathsf{K}\psi$ and $(a, s_2) \Vdash \mathsf{K}\psi$ are both equivalent to the statement $\forall s \in \{s_1, s_2\}((a, s) \Vdash \psi)$, by item 5 of Definition 2 and because $s_1 \sim_a s_2$. Then, statements $(a, s_1) \Vdash \mathsf{K}\psi$ and $(a, s_2) \Vdash \mathsf{K}\psi$ are equivalent.

If formula $\varphi$ has the form $\mathsf{D}\psi$, then, by item 6 of Definition 2 and because the model has a single name *Ann*, the statements $(a, s_1) \Vdash \varphi$ and $(a, s_2) \Vdash \varphi$ are both equivalent to the conjunction of the statements $(a, s_1) \Vdash \psi$ and $(a, s_2) \Vdash \psi$.

Suppose formula $\varphi$ has the form $\mathsf{R}\psi$. Note that the statement $(a, s_1) \Vdash \mathsf{R}\psi$ and the statement $(a, s_2) \Vdash \mathsf{R}\psi$ are both equivalent to the statement

$$\exists a' \forall s \in \{s_1, s_2\}((a', s) \Vdash \psi),$$

by item 6 of Definition 2 and because $s_1 \sim_a s_2$. Thus, statements $(a, s_1) \Vdash \mathsf{R}\psi$ and $(a, s_2) \Vdash \mathsf{R}\psi$ are equivalent. $\boxtimes$

**Lemma 7** *$(a, s_1) \Vdash \mathsf{A}p$ and $(a, s_2) \nVdash \mathsf{A}p$.*

Proof. Note that $(b, s_1) \Vdash p$ and $(b, s_2) \nVdash p$, see Figure 4. Thus, $(a, s_1) \Vdash \mathsf{A}p$ and $(a, s_2) \nVdash \mathsf{A}p$ by item 4 of Definition 2. $\boxtimes$

The next result follows from the two previous lemmas:

**Theorem 3** *Modality $\mathsf{A}$ is not definable in language $\Phi^{\mathsf{K},\mathsf{D},\mathsf{R}}$.*

### 8.4 Undefinability of $\mathsf{D}$

To prove the undefinability of modality $\mathsf{D}$ in language $\Phi^{\mathsf{A},\mathsf{K},\mathsf{R}}$, consider two Grove-Halpern models both of which have a single state $s$ and two agents, $a$ and $b$, present in the state $s$. Let $\pi(p)$ be the singleton $\{(a,s)\}$ in both models. The set of names in both models consists of a single name *Ann*. In the first model, each agent uses this name to refer to agent $a$. In the second model, each agent uses this name to refer to agent $b$.

The next lemma holds because definitions of modalities $\mathsf{A}$, $\mathsf{K}$, and $\mathsf{R}$ (see items 4, 5, and 7 of Definition 2) refer neither to the names nor to the identification mechanism.

**Lemma 8** *For any formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{R}}$, statement $(a,s) \Vdash \varphi$ is true in the first model iff it is true in the second model.*

The next lemma holds by item 6 of Definition 2 because agent $a$ can refer to itself (as *Ann*) in the first model, but there is no way to do this in the second model.

**Lemma 9** *Statement $(a,s) \Vdash \mathsf{D}p$ is true in the first model and false in the second model.*

The next result follows from the two previous lemmas:

**Theorem 4** *Modality $\mathsf{D}$ is not definable in language $\Phi^{\mathsf{A},\mathsf{K},\mathsf{R}}$.*

## 9 Logic of De Dicto Knowing Who

In the rest of this article, we give a sound and complete axiomatization of the interplay between modalities $\mathsf{D}$, $\mathsf{K}$, and $\mathsf{A}$.

### 9.1 Axioms

In addition to propositional tautologies in language $\Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$, our logical system has the following axioms, where the symbol $\Box$ is either modality $\mathsf{A}$ or modality $\mathsf{K}$ in the rest of this article:

1. Truth: $\Box\varphi \to \varphi$,
2. Distributivity: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
3. Negative Introspection: $\neg\Box\varphi \to \Box\neg\Box\varphi$,
4. Know-Nobody: $\mathsf{A}\neg\varphi \to \neg\mathsf{D}\varphi$,
5. Know-All: $\mathsf{KA}(\varphi \to \psi) \to (\mathsf{D}\varphi \to \mathsf{D}\psi)$,
6. Introspection of Knowing-Who: $\mathsf{D}\varphi \to \mathsf{KD}\varphi$.

The Truth, the Distributivity, and the Negative Introspection are standard S5 axioms. The Know-Nobody axiom says that if there is no agent in the current state for whom $\varphi$ is true, then the current agent cannot know somebody for whom $\varphi$ is true. The Know-All axiom says that if in the current state the agent knows that $\varphi \to \psi$ for all agents and she also knows that $\varphi$ is true for some agent, then she knows someone for whom $\psi$ is true. The Introspection of Knowing-Who axiom says that if the current agent knows for whom $\varphi$ is true, then the agent knows that it knows.

We write $\vdash \varphi$ and say that formula $\varphi$ is a *theorem* of our logical system if formula $\varphi$ is provable in our logical system using the Modus Ponens inference rule and the three forms of the Necessitation inference rule:

$$\frac{\varphi, \quad \varphi \to \psi}{\psi} \qquad \frac{\varphi}{\mathsf{A}\varphi} \qquad \frac{\varphi}{\mathsf{K}\varphi} \qquad \frac{\varphi}{\mathsf{D}\varphi}.$$

In addition to unary relation $\vdash \varphi$, we also consider a binary relation $X \vdash \varphi$ between a set of formulae $X \subseteq \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$ and a formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$. Let $X \vdash \varphi$ if formula $\varphi$ is provable from the *theorems* of our logical system and the set of additional formulae $X$ using *only* the Modus Ponens inference rule. Note that statement $\varnothing \vdash \varphi$ is equivalent to $\vdash \varphi$. We say that set $X$ is consistent if there is no formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$ such that $X \vdash \varphi$ and $X \vdash \neg\varphi$.

Next, we state two well-known facts about S5 modality that will be used later. We reproduce their proofs here to keep our presentation self-contained.

**Lemma 10** *If $\varphi_1, \dots, \varphi_n \vdash \psi$, then $\Box\varphi_1, \dots, \Box\varphi_n \vdash \Box\psi$.*

PROOF. Suppose that $\varphi_1, \dots, \varphi_n \vdash \psi$. Thus, by the deduction lemma for propositional logic applied $n$ times, $\vdash \varphi_1 \to (\varphi_2 \to \dots (\varphi_n \to \psi) \dots)$. Hence, by the Necessitation inference rule, $\vdash \Box(\varphi_1 \to (\varphi_2 \to \dots (\varphi_n \to \psi) \dots))$. Then, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \Box\varphi_1 \to \Box(\varphi_2 \to \dots (\varphi_n \to \psi) \dots)$. Thus, $\Box\varphi_1 \vdash \Box(\varphi_2 \to \dots (\varphi_n \to \psi) \dots)$, again by the Modus Ponens inference rule. Therefore, by repeating the previous two steps $(n-1)$ more times, $\Box\varphi_1, \dots, \Box\varphi_n \vdash \Box\psi$. $\boxtimes$

To keep our work self-contained, we reproduce the proof of the following well-known result in the appendix.

**Lemma 11** $\vdash \Box\varphi \to \Box\Box\varphi$.

## 9.2 Soundness

In this subsection, we prove the soundness of our logical system. The soundness of the Truth, the Distributivity, and the Negative Introspection axioms, as well as the Modus Ponens and the three forms of the Necessitation inference rule, is straightforward. Below we show the soundness of each remaining axiom as a separate lemma. In these lemmas we assume that $(a, s)$ is an arbitrary pair of an agent $a$ and a state $s$ such that $s \in P(a)$.

**Lemma 12** *If* $(a, s) \Vdash \mathsf{A}\neg\varphi$*, then* $(a, s) \nVdash \mathsf{D}\varphi$*.*

PROOF. Suppose that $(a, s) \Vdash \mathsf{D}\varphi$. Thus, by item 6 of Definition 2, there is a name $n \in N$ such that for each state $s' \in P(a)$ and each agent $a' \in A$, if $s \sim_a s'$ and $(a, s', n, a') \in I$, then $(a', s') \Vdash \varphi$.

Note that $s \in P(a)$ by the assumption in the preamble of this section and $s \sim_a s$ because $\sim_a$ is an equivalence relation. Thus, for each agent $a' \in A$, if $(a, s, n, a') \in I$, then $(a', s) \Vdash \varphi$.

By condition (a) of item 6 in Definition 1, there is at least one agent $a' \in A$ such that $(a, s, n, a') \in I$. Thus, $(a', s) \Vdash \varphi$. Hence, $(a', s) \nVdash \neg\varphi$ by item 2 of Definition 2. Therefore, $(a, s) \nVdash \mathsf{A}\neg\varphi$ by item 4 of Definition 2. ⊠

**Lemma 13** *If* $(a, s) \Vdash \mathsf{KA}(\varphi \to \psi)$ *and* $(a, s) \Vdash \mathsf{D}\varphi$*, then* $(a, s) \Vdash \mathsf{D}\psi$*.*

PROOF. Suppose that $(a, s) \Vdash \mathsf{D}\varphi$. Thus, by item 6 of Definition 2, there is a name $n \in N$ such that for each state $s' \in P(a)$ and each agent $a' \in A$, if $s \sim_a s'$ and $(a, s', n, a') \in I$, then $(a', s') \Vdash \varphi$.

Consider any state $s' \in P(a)$ and any agent $a' \in A$ such that $s \sim_a s'$ and $(a, s', n, a') \in I$. Then, as we have shown above,

$$(a', s') \Vdash \varphi. \tag{2}$$

By item 6 of Definition 2, it will suffice to show that $(a', s') \Vdash \psi$. Indeed, by item 5 of Definition 2 assumption $(a, s) \Vdash \mathsf{KA}(\varphi \to \psi)$ implies that $(a, s') \Vdash \mathsf{A}(\varphi \to \psi)$ because $s' \in P(a)$ and $s \sim_a s'$.

Note that $s' \in P(a')$ by condition (b) of item 6 in Definition 1 because $(a, s', n, a') \in I$. Hence, statement $(a, s') \Vdash \mathsf{A}(\varphi \to \psi)$ implies that $(a', s') \Vdash \varphi \to \psi$ by item 4 of Definition 2. Therefore, $(a', s') \Vdash \psi$ by item 3 of Definition 2 and statement (2). ⊠

**Lemma 14** *If* $(a, s) \Vdash \mathsf{D}\varphi$*, then* $(a, s) \Vdash \mathsf{KD}\varphi$*.*

PROOF. Consider any state $s' \in P(a)$ such that $s \sim_a s'$. By item 5 of Definition 2, it suffices to show that $(a, s') \Vdash \mathsf{D}\varphi$.

By item 6 of Definition 2, assumption $(a, s) \Vdash \mathsf{D}\varphi$ implies that there is a name $n \in N$ such that for each state $s'' \in P(a)$ and each agent $a' \in A$, if $s \sim_a s''$ and $(a, s'', n, a') \in I$, then $(a', s'') \Vdash \varphi$. Recall that $s \sim_a s'$. Thus, for each state $s'' \in P(a)$ and each agent $a' \in A$, if $s' \sim_a s''$ and $(a, s'', n, a') \in I$, then $(a', s'') \Vdash \varphi$ because $\sim_a$ is an equivalence relation. Therefore, $(a, s') \Vdash \mathsf{D}\varphi$ by item 6 of Definition 2. ⊠

## 9.3 Completeness

In the rest of this article, we prove the completeness of our logical system. We start by highlighting the key steps in the proof of the completeness.

### 9.3.1 Overview

In modal logic, a completeness proof usually constructs a canonical model with states being maximal consistent sets. The key property of the canonical model is normally captured by the "induction" or "truth" lemma which ordinarily states that a formula is satisfied at a state if and only if it belongs to the corresponding maximal consistent set. In our case, the satisfiability is defined as a relation $(a, s) \Vdash \varphi$ between an agent $a$, a state $s$, and a formula $\varphi$. As a result, in our construction, a maximal consistent set corresponds not to a state, but to a pair $(a, s)$ consisting of an agent $a$ and a state $s$. We informally refer to such pairs as "views". The induction lemma in our article is Lemma 29. It states that a formula is satisfied at a view if and only if it belongs to the maximal consistent set corresponding to this view.

There are three distinct challenges that we faced while proving the completeness theorem. The first of them is how to define agents and states, assuming that views are maximal consistent sets of formulae. Our first attempt was based on the observation that two views that have the same states satisfy exactly the same $\mathsf{A}$-formulae. Thus, one can define states as classes of views (maximal consistent sets) that have the same $\mathsf{A}$-formulae. Similarly, it is reasonable to assume that if two sets have exactly the same $\mathsf{K}$-formulae, then they correspond to two views of the same agent in two indistinguishable states. Hence, one can define agents as classes of views that have the same $\mathsf{K}$-formulae. The problem with this approach is that there could be two distinct maximal consistent sets that have the same $\mathsf{A}$-formulae and the same $\mathsf{K}$-formula. Such sets could be unequal because, for example, one of them contains a propositional variable and the other the negation of the same variable. Informally, such sets would correspond to two different views of the same agent in the same state. This is problematic because our formal semantics captured in Definition 2 assumes that if an agent $a$ is present in a state $s$, then the agent has a unique view $(a, s)$ in this state.

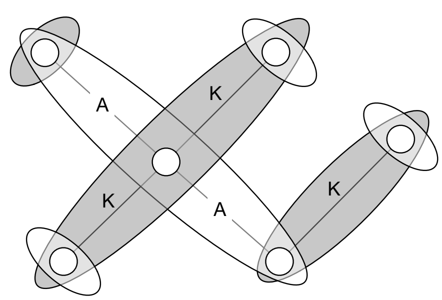


Figure 5: Nodes are views, white $\mathsf{A}$-classes are states, and grey $\mathsf{K}$-classes are agents. There are 4 states and 3 agents depicted in the diagram.

To solve this problem, we need to guarantee that any class of views representing a state has at most one common element with any class of views representing an agent. We achieve this by using a *tree construction*. The canonical model in our proof is a tree whose nodes are labeled with maximal consistent sets and edges are labeled with a single modality: either $\mathsf{A}$ or $\mathsf{K}$, see Figure 5. Informally, nodes of this tree correspond to views. We say that two nodes are $\mathsf{A}$-equivalent if all edges along the simple path between these two nodes are labeled with modality $\mathsf{A}$ and define states as equivalence classes with respect to this relation. Similarly, nodes are $\mathsf{K}$-equivalent if all edges along the simple path between them are labeled with modality $\mathsf{K}$. Agents are $\mathsf{K}$-equivalence classes of nodes. Note that there is a unique simple path between any two nodes in a tree. As a result, the same two nodes cannot be $\mathsf{A}$-equivalent and $\mathsf{K}$-equivalent at the same time. Thus, this construction results in at most one node (view) corresponding to any pair consisting of an agent and a state. This guarantees that there is at most one view for any agent in any state. Of course, an agent ($\mathsf{K}$-equivalence class) might have no common nodes with a state ($\mathsf{A}$-equivalence class). In this case, the agent is not present in the state.

As pointed out earlier, any two views that have the same state must have the same $\mathsf{A}$-formulae. We guarantee this by requiring any two nodes connected by an $\mathsf{A}$-edge to have the same $\mathsf{A}$-formulae. Similarly, we require any two nodes connected by a $\mathsf{K}$-edge to have the same $\mathsf{K}$-formulae.

Trees have previously been used in the proof of completeness of distributed knowledge [8]. The use of trees to guarantee that intersections of classes of nodes have at most one element is an original contribution of this work.

The second major challenge that we had to overcome while proving the completeness is creating the actual nodes, or maximal consistent sets of formulae. The standard proof of completeness in modal logic usually contains a "child" lemma that for each maximal consistent set $X$ and each formula $\neg\Box\varphi \in X$ constructs another set that contains formula $\neg\varphi$. In our case, these are Lemma 26 and Lemma 27 for modalities $\mathsf{K}$ and $\mathsf{A}$ respectively. The situation is more complicated for modality $\mathsf{D}$ because one needs to construct two new interdependent maximal consistent sets simultaneously: one that corresponds to view $(a, s')$ and another to view $(a', s')$, see item 6 of Definition 2. Unfortunately, because of the interdependency, these two sets cannot be constructed consecutively. To construct them simultaneously, we developed a new technique that consists in defining a property of a pair of sets of formulae, choosing a pair of small sets satisfying this property, and then extending the sets while maintaining the property. When fully extended, each of the sets will become the label of a node in the tree construction that we described above and will represent a view in our model. Informally, the property that we maintain could be described as "views can co-exist in the same states". We call such views *consonant*. A somewhat similar construction of two interdependent nodes has been used in [18, 17] to construct two states of a game in "harmony". The construction proposed in this article creates two nodes that belong to the same state and, thus have the same $\mathsf{A}$-formulae. The two states in "harmony" are consecutive states of a game that do not share any specific class of formulae. As a result, the properties of consonant pairs are different from the properties of pairs in "harmony" and the proofs that the corresponding constructions work are also different.

The third challenge in constructing the canonical model is to define the right identification mechanism. What name should one of the grey classes (agents) in Figure 5 use to refer to another grey class in one of the white classes (states)? The first idea that we considered was unbelievably simple. If $(a, s) \Vdash \mathsf{D}\varphi$, then formula $\varphi$ *itself* could be the name under which agent $a$ knows the agent with property $\varphi$ in state $s$. Although elegant, this naming scheme has a fatal flaw: it does not distinguish between knowing that an agent *exists* and *knowing who* the agent is. This namespace makes the formula $\mathsf{K}\neg\mathsf{A}\neg\varphi \to \mathsf{D}\varphi$ true in any model that uses this identification mechanism. Since this formula is not universally valid, the mechanism cannot be used in the canonical model construction of the completeness proof.

We solved this problem by modifying the above identification mechanism. In the settings of Figure 5, if the maximal consistent set of the unique node at the intersection of an agent $a$ and a state $s$ contains formula $\mathsf{D}\varphi$, then name $\varphi$, when used by agent $a$ at state $s$, refers to each agent $b$ present in the state $s$ such that the maximal consistent set of the unique node at the intersection of agent $b$ and state $s$ contains formula $\varphi$. Otherwise, name $\varphi$ refers to all agents present in state $s$.

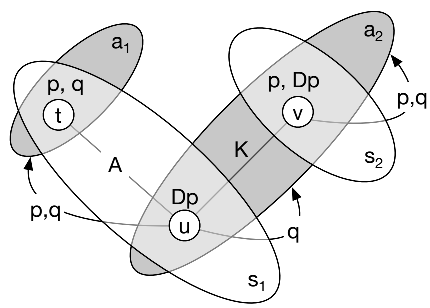


Figure 6: Fragment of a Canonical Model.

As an example, consider the fragment of the tree depicted in Figure 6. Nodes $u$ and $t$ are connected by an $\mathsf{A}$-edge. Thus, they represent views of two different agents, $a_1$ and $a_2$, in the same state $s_1$. On the other hand, nodes $u$ and $v$ are connected by a $\mathsf{K}$-edge. Hence, they represent two views of the same agent $a_2$ in two different (but indistinguishable to the agent) states: $s_1$ and $s_2$. Note that agent $a_1$ is not present in state $s_2$ because the corresponding ovals have no common nodes. The maximal consistent sets associated with views $u$ and $v$ contain the formula $\mathsf{D}p$. As a result, when name $p$ is used in these two views, it refers to the agents in the same state whose maximal consistent sets contain variable $p$. In other words, when name $p$ is used by agent $a_2$ in state $s_1$, it refers to agent $a_1$ and when the same name is used by the same agent in state $s_2$, it refers to agent $a_2$ itself. At the same time, because formula $\mathsf{D}q$ does not belong to the maximal consistent sets corresponding to nodes $u$ and $v$, when name $q$ is used in these two views, it refers to all agents present in the corresponding state. In other words, in state $s_1$ name $q$ is used by agent $a_2$ to refer to itself and agent $a_1$; in state $s_2$ the same name is used by the same agent to refer only to itself.

This concludes the overview of the proof of the strong completeness theorem. We divided the proof into several sections. First, we define the tree structure of a canonical model. Then, we define the canonical model itself. In the section that follows, we introduce the notion of a consonant pair of maximal consistent sets of formulae and prove its properties that are required for the proof of completeness. After that, we prove several properties of the canonical model, one of the proofs is using consonant pairs. We finish by proving the completeness theorem.

### 9.3.2 Tree Construction

In our informal earlier discussion of the canonical model, we said that "views" are represented in the canonical model by nodes of a tree whose edges are labeled with modalities $\mathsf{A}$ and $\mathsf{K}$ and whose nodes are labeled with maximal consistent sets, see Figure 7. Mathematically, such nodes could be defined by a sequence of labels along a path from the root of the tree to the node. This is the approach that we take in the definition below.

**Definition 3** *Let $\Omega$ be the set of finite sequences $X_0, \Box_1, X_1, \Box_2, \dots, \Box_n, X_n$ such that $n \geq 0$ and*

1. *$X_1, \dots, X_n$ are maximal consistent sets of formulae,*
2. *$\Box_1, \dots, \Box_n \in \{\mathsf{A}, \mathsf{K}\}$ are modalities,*
3. *$\{\varphi \mid \Box_i\varphi \in X_{i-1}\} \subseteq X_i$ for each integer $i \geq 1$.*

If $\omega$ is a sequence $X_0, \Box_1, X_1, \Box_2, \dots, \Box_n, X_n$, then by $hd(\omega)$ we mean set $X_n$. For any finite sequence $y = y_1, \dots, y_n$ and any element $z$, by $y :: z$ we mean the sequence $y_1, \dots, y_n, z$.

Next, we show that any two views connected by an $\mathsf{A}$-edge share $\mathsf{A}$-formulae and any two views connected by a $\mathsf{K}$-edge share $\mathsf{K}$-formulae.

**Lemma 15** *$\Box_i\varphi \in X_{i-1}$ iff $\Box_i\varphi \in X_i$ for each sequence $X_0, \Box_1, X_1, \Box_2, \dots, \Box_n, X_n \in \Omega$, each integer $i \geq 1$, and each formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.*

Proof. $(\Rightarrow)$ : Suppose $\Box_i\varphi \in X_{i-1}$. Hence, by Lemma 11, it follows that $X_{i-1} \vdash \Box_i\Box_i\varphi$. Thus, $\Box_i\Box_i\varphi \in X_{i-1}$ because set $X_i$ is maximal. Therefore, $\Box_i\varphi \in X_i$ by item 3 of Definition 3.

$(\Leftarrow)$ : Assume that $\Box_i\varphi \notin X_{i-1}$. Hence, $\neg\Box_i\varphi \in X_{i-1}$ because set $X_{i-1}$ is maximal. Thus, $X_{i-1} \vdash \Box_i\neg\Box_i\varphi$ by the Negative Introspection axiom and the Modus Ponens inference rule. Then, $\Box_i\neg\Box_i\varphi \in X_{i-1}$ because set $X_{i-1}$ is maximal. Hence, $\neg\Box_i\varphi \in X_i$ by item 3 of Definition 3. Therefore, $\Box_i\varphi \notin X_i$ because set $X_i$ is consistent. $\boxtimes$

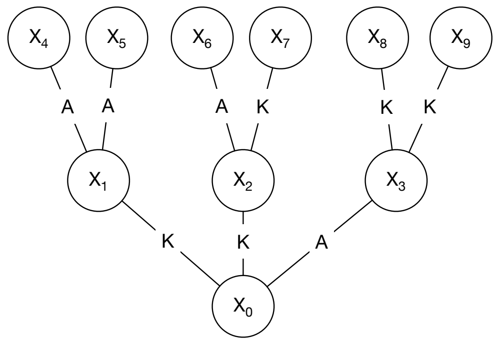


Figure 7: Fragment of the tree formed by worlds of a canonical model.

We say that sequences $\omega \in \Omega$ and $\omega' \in \Omega$ are *adjacent* if $\omega' = \omega :: \Box :: X$ for some $\Box \in \{\mathsf{A}, \mathsf{K}\}$ and some maximal consistent set $X$. The adjacency relation defines a tree (undirected connected graph without cycles) structure on sets $\Omega$. The elements of set $\Omega$ will be called the *nodes* of this tree. If $\omega' = \omega :: \Box :: X$, then we say that undirected edge $(\omega, \omega')$ is *labeled* with modality $\Box$. It is convenient to visualize node $\omega \in \Omega$ to be labeled with the set $hd(\omega)$, see Figure 7. Recall that there is a unique simple path between any two nodes of a tree. This path is used in Definition 4, Definition 5, and the key Lemma 18 below.

**Definition 4** *$\omega \overset{\mathsf{S}}{\sim} \omega'$ if each edge along the unique simple path between nodes $\omega$ and $\omega'$ is labeled with modality $\mathsf{A}$.*

Note that the relation $\overset{\mathsf{S}}{\sim}$ is an equivalence relation on $\Omega$. In the next subsection, we define states of a canonical model as equivalence classes of this relation.

**Lemma 16** *If $\omega \overset{\mathsf{S}}{\sim} \omega'$, then $\mathsf{A}\varphi \in hd(\omega)$ iff $\mathsf{A}\varphi \in hd(\omega')$ for each formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.*

PROOF. The statement of the lemma follows from Definition 4 and Lemma 15. $\boxtimes$

**Definition 5** *$\omega \overset{\mathsf{A}}{\sim} \omega'$ if each edge along the unique simple path between nodes $\omega$ and $\omega'$ is labeled with modality $\mathsf{K}$.*

Note that the relation $\overset{\mathsf{A}}{\sim}$ is an equivalence relation on $\Omega$. In the next subsection, we define agents of a canonical model as equivalence classes of this relation.

**Lemma 17** *If* $\omega \overset{\mathsf{A}}{\sim} \omega'$*, then* $\mathsf{K}\varphi \in hd(\omega)$ *iff* $\mathsf{K}\varphi \in hd(\omega')$ *for each formula* $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.

PROOF. The statement of the lemma follows from Definition 5 and Lemma 15. ⊠

#### 9.3.3 Canonical Model

For any maximal consistent set of formulae $X_0$, we define canonical model $M(X_0) = (S, A, P, \{\sim_a\}_{a\in A}, N, I, \pi)$.

The states of our model are equivalence classes of $\Omega$ with respect to the equivalence relation $\overset{\mathsf{S}}{\sim}$. The set of names is the set of formulae. Thus, $S = \Omega/\overset{\mathsf{S}}{\sim}$ and $N = \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.

The agents are equivalence classes of $\Omega$ with respect to the equivalence relation $\overset{\mathsf{A}}{\sim}$. That is $A = \Omega/\overset{\mathsf{A}}{\sim}$. For any sequence $\omega \in \Omega$, by $[\omega]_{\mathsf{S}}$ and $[\omega]_{\mathsf{A}}$ we denote the equivalence classes of $\omega$ with respect to equivalence relations $\overset{\mathsf{S}}{\sim}$ and $\overset{\mathsf{A}}{\sim}$ accordingly.

**Definition 6** $[\sigma]_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ *if* $[\alpha]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}} \neq \varnothing$.

As we discussed in Subsection 9.3.1, the purpose of the tree construction is to guarantee that each agent has at most one view in each state. The next lemma shows that the tree construction achieves this goal.

**Lemma 18** *The set* $[\alpha]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}}$ *contains at most one element.*

PROOF. Consider any nodes $\theta_1, \theta_2 \in [\alpha]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}}$. Then, $\theta_1, \theta_2 \in [\alpha]_{\mathsf{A}}$ and $\theta_1, \theta_2 \in [\sigma]_{\mathsf{S}}$. Thus, $\theta_1 \overset{\mathsf{A}}{\sim} \theta_2$ and $\theta_1 \overset{\mathsf{S}}{\sim} \theta_2$. By Definition 5, statement $\theta_1 \overset{\mathsf{A}}{\sim} \theta_2$ implies that each edge along the unique simple path between nodes $\theta_1$ and $\theta_2$ is labeled with $\mathsf{K}$. Similarly, by Definition 4, statement $\theta_1 \overset{\mathsf{S}}{\sim} \theta_2$ implies that each edge along the unique simple path between nodes $\theta_1$ and $\theta_2$ is labeled with $\mathsf{A}$. Therefore, $\theta_1 = \theta_2$ because each edge has only one label. ⊠

Informally, by $\theta(a, s)$ we denote the unique view of agent $a$ in state $s$, if such a view exists.

**Definition 7** *If* $[\alpha]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}} \neq \varnothing$*, then* $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})$ *is the unique node of the set* $[\alpha]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}}$.

We are ready to continue the definition of model $M(X_0)$. Informally, we say that agent $a$ cannot distinguish states $s_1$ and $s_2$ if the unique simple path connecting nodes $\theta(a, s_1)$ and $\theta(a, s_2)$ is labeled by $\mathsf{A}$ only.

**Definition 8** *For any* $[\alpha]_{\mathsf{A}} \in A$*,* $[\sigma_1]_{\mathsf{S}}, [\sigma_2]_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$*, let* $[\sigma_1]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma_2]_{\mathsf{S}}$ *if* $\theta([\alpha]_{\mathsf{A}}, [\sigma_1]_{\mathsf{S}}) \overset{\mathsf{A}}{\sim} \theta([\alpha]_{\mathsf{A}}, [\sigma_2]_{\mathsf{S}})$.

Note that this means that in the canonical model an agent cannot distinguish any two states in which the agent itself is present.

**Lemma 19** *Relation $\sim_{[\alpha]_\mathsf{A}}$ is an equivalence relation on set $P([\alpha]_\mathsf{A})$.* $\boxtimes$

Next, we define the identification mechanism of the canonical model. Recall from Subsection 9.3.1 section that if the maximal consistent set of node $\theta(a,s)$ contains formula $\mathsf{D}\varphi$, then name $\varphi$, when used by agent $a$ at state $s$, refers to all agents $b$ present in state $s$ such that the maximal consistent set of the node $\theta(b,s)$ contains formula $\varphi$. Otherwise, name $\varphi$ refers to all agents present in state $s$.

**Definition 9** *Relation $I \subseteq A\times S\times N\times A$ is a set of tuples $([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S},\varphi,[\alpha_2]_\mathsf{A})$ such that*

1. *$[\sigma]_\mathsf{S}\in P([\alpha_1]_\mathsf{A})$, $[\sigma]_\mathsf{S}\in P([\alpha_2]_\mathsf{A})$, and*
2. *if $\mathsf{D}\varphi\in hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))$, then $\varphi\in hd(\theta([\alpha_2]_\mathsf{A},[\sigma]_\mathsf{S}))$.*

The next lemma proves item 6(a) of Definition 1. Note that the condition $[\sigma]_\mathsf{S}\in P([\alpha_2]_\mathsf{A})$ in item 1 of Definition 9 guarantees item 6(b) of Definition 1.

**Lemma 20** *For each agent $[\alpha_1]_\mathsf{A}\in A$, each state $[\sigma]_\mathsf{S}\in P([\alpha_1]_\mathsf{A})$, and each name $\varphi\in N$, there is an agent $[\alpha_2]_\mathsf{A}\in A$ such that $([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S},\varphi,[\alpha_2]_\mathsf{A})\in I$.*

PROOF. Assumption $[\sigma]_\mathsf{S}\in P([\alpha_1]_\mathsf{A})$ implies existence of node $\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S})$ by Definition 6 and Definition 7. We consider the following two cases separately:
**Case I**: $\mathsf{D}\varphi\notin hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))$. Then, take agent $[\alpha_2]_\mathsf{A}$ to be agent $[\alpha_1]_\mathsf{A}$. Thus, $([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S},\varphi,[\alpha_2]_\mathsf{A})\in I$ by Definition 9, assumption $[\sigma]_\mathsf{S}\in P([\alpha_1]_\mathsf{A})$ of the lemma, and assumption $\mathsf{D}\varphi\notin hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))$ of the case.
**Case II**: $\mathsf{D}\varphi\in hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))$. Consider the set of formulae

$$X=\{\varphi\}\cup\{\psi\mid \mathsf{A}\psi\in hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))\}.$$

**Claim 1** *Set $X$ is consistent.*

PROOF OF CLAIM. Suppose the opposite. Thus, there are formulae

$$\mathsf{A}\psi_1,\dots,\mathsf{A}\psi_n\in hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S})) \tag{3}$$

such that $\psi_1,\dots,\psi_n\vdash\neg\varphi$. Hence, $\mathsf{A}\psi_1,\dots,\mathsf{A}\psi_n\vdash\mathsf{A}\neg\varphi$ by Lemma 10. Then, $hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))\vdash\mathsf{A}\neg\varphi$ by statement (3). Thus,

$$hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))\vdash\neg\mathsf{D}\varphi$$

by the Know-Nobody axiom and the Modus Ponens inference rule. Hence, $\mathsf{D}\varphi\notin hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))$ because set $hd(\theta([\alpha_1]_\mathsf{A},[\sigma]_\mathsf{S}))$ is consistent, which contradicts the assumption of the case. Therefore, set $X$ is consistent. This concludes the proof of the claim. $\boxtimes$

Let $X'$ be any maximal consistent extension of set $X$ and $\alpha_2=\sigma::\mathsf{A}::X'$.

**Claim 2** *$\alpha_2 \in \Omega$ and $\alpha_2 \in [\alpha_2]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}}$.*

Proof of Claim. We prove the two statements separately.

Consider any formula $\mathsf{A}\psi \in hd(\sigma)$. By Definition 3, it suffices to show that $\psi \in X'$. Indeed, $\sigma \in [\sigma]_{\mathsf{S}}$ because $[\sigma]_{\mathsf{S}}$ is an equivalence class and $\theta([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \in [\sigma]_{\mathsf{S}}$ by Definition 7. Thus, $\sigma \sim_{\mathsf{S}} \theta([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})$. Hence, by Lemma 16, $\mathsf{A}\psi \in hd(\theta([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Then, $hd(\theta([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})) \vdash \psi$ by the Truth axiom and the Modus Ponens inference rule. Hence, $\psi \in hd(\theta([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ because set $hd(\theta([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal. Thus, $\psi \in X \subseteq X'$ by the choice of sets $X$ and $X'$.

Note that $\alpha_2 \in [\alpha_2]_{\mathsf{A}}$ because $[\alpha_2]_{\mathsf{A}}$ is an equivalence class. To prove $\alpha_2 \in [\sigma]_{\mathsf{S}}$, recall that $\alpha_2 = \sigma :: \mathsf{A} :: X'$. Thus, $\alpha_2 \overset{\mathsf{S}}{\sim} \sigma$ by Definition 4. Therefore, $\alpha_2 \in [\sigma]_{\mathsf{S}}$. ⊠

Note that $[\sigma]_{\mathsf{S}} \in P([\alpha_1]_{\mathsf{A}})$ by the assumption of the lemma. In addition, $[\sigma]_{\mathsf{S}} \in P([\alpha_2]_{\mathsf{A}})$ by Definition 6 and Claim 2. Also, $\alpha_2 = \theta([\alpha_2]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})$ by Definition 7 and Claim 2. Hence, $\varphi \in X \subseteq X' = hd(\alpha_2) = hd(\theta([\alpha_2]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$, by the choice of set $X$, the choice of set $X'$, and the choice of sequence $\alpha_2$. Thus, $([\alpha_1]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}, \varphi, [\alpha_2]_{\mathsf{A}}) \in I$ by Definition 9.

This concludes the proof of Lemma 20. ⊠

**Definition 10** *$\pi(p)$ is the set of all pairs $([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})$ such that $[\sigma]_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ and $p \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$.*

### 9.3.4 Consonant Pair

Recall from the Overview subsection that the purpose of the consonant pairs is to simultaneously construct two interdependent maximal consistent sets of formulae. We will start with an "initial" pair of sets, prove that this pair is consonant and then we will extend the sets in the pair while maintaining the pair being consonant. By $\bigwedge X$ we mean the conjunction of all formulae in set $X$. As usual, $\bigwedge \varnothing$ is constant $\top$.

**Definition 11** *A pair $(X, Y)$ of sets of formulae is consonant if $X \nvdash \mathsf{A}\neg \bigwedge Y'$ for each finite set $Y' \subseteq Y$.*

**Lemma 21** *If pair $(X, Y)$ is consonant, then sets $X$ and $Y$ are consistent.*

Proof. Assume that set $X$ is not consistent. Thus, $X \vdash \varphi$ for each formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$. In particular, $X \vdash \mathsf{A}\neg \bigwedge \varnothing$. Therefore, pair $(X, Y)$ is not consonant by Definition 11 and because $\varnothing \subseteq Y$.

Let us now suppose that set $Y$ is not consistent. Thus, there is a finite subset $Y' \subseteq Y$ such that $\vdash \neg \bigwedge Y'$. Hence, $\vdash \mathsf{A}\neg \bigwedge Y'$ by the Necessitation inference rule. Therefore, pair $(X, Y)$ is not consonant by Definition 11. ⊠

Next, we show that a certain "initial" pair is consonant.

**Lemma 22** *The pair* $(\{\psi \mid \mathsf{K}\psi \in Z\}, \{\neg\varphi, \varphi'\})$ *is consonant for any consistent set of formula* $Z$ *and any formulae* $\neg\mathsf{D}\varphi, \mathsf{D}\varphi' \in Z$.

Proof. Suppose the opposite. Thus, by Definition 11, there are formulae

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_n \in Z \tag{4}$$

and a finite set of formulae $Y \subseteq \{\neg\varphi, \varphi'\}$ such that

$$\psi_1, \dots, \psi_n \vdash \mathsf{A}\neg \bigwedge Y. \tag{5}$$

At the same time, note that formula $\neg \bigwedge Y \to \neg(\varphi' \bigwedge \neg\varphi)$ is a propositional tautology because $Y \subseteq \{\neg\varphi, \varphi'\}$. Thus, formula $\neg \bigwedge Y \to (\varphi' \to \varphi)$ is also a propositional tautology. Hence, $\vdash \mathsf{A}(\neg \bigwedge Y \to (\varphi' \to \varphi))$ by the Necessitation inference rule. Then, $\psi_1, \dots, \psi_n \vdash \mathsf{A}(\varphi' \to \varphi)$ by the Distributivity axiom and the Modus Ponens rule using statement (5). Thus, $\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_n \vdash \mathsf{KA}(\varphi' \to \varphi)$ by Lemma 10. Hence, $Z \vdash \mathsf{KA}(\varphi' \to \varphi)$ because of statement (4). Then, $Z \vdash \mathsf{D}\varphi' \to \mathsf{D}\varphi$ by the Know-All axiom and the Modus Ponens inference rule. Thus, $Z \vdash \mathsf{D}\varphi$ by the Modus Ponens inference rule and the assumption $\mathsf{D}\varphi' \in Z$ of the lemma. Therefore, $\neg\mathsf{D}\varphi \notin Z$ because set $Z$ is consistent, which contradicts the assumption $\neg\mathsf{D}\varphi \in Z$ of the lemma. ⊠

Now, we prove that any consonant pair could be "extended" in a certain way to still be consonant.

**Lemma 23** *If pair* $(X, Y)$ *is consonant and* $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$ *is an arbitrary formula, then either pair* $(X \cup \{\neg\mathsf{A}\varphi\}, Y)$ *or pair* $(X, Y \cup \{\varphi\})$ *is consonant.*

Proof. Let pairs $(X \cup \{\neg\mathsf{A}\varphi\}, Y)$ and $(X, Y \cup \{\varphi\})$ be not consonant. Thus, by Definition 11, there are finite sets $Y', Y'' \subseteq Y$ such that

$$X, \neg\mathsf{A}\varphi \vdash \mathsf{A}\neg \bigwedge Y' \tag{6}$$

and, for some $Z \subseteq \{\varphi\} \cup Y''$,

$$X \vdash \mathsf{A}\neg \bigwedge Z. \tag{7}$$

Observe that formula $\neg \bigwedge Z \to (\varphi \to \neg \bigwedge(Y' \cup Y''))$ is a tautology because $Z \subseteq \{\varphi\} \cup Y'' \subseteq \{\varphi\} \cup Y' \cup Y''$. Hence,

$$\vdash \mathsf{A}(\neg \bigwedge Z \to (\varphi \to \neg \bigwedge(Y' \cup Y'')))$$

by the Necessitation rule. Thus, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{A}\neg \bigwedge Z \to \mathsf{A}(\varphi \to \neg \bigwedge(Y' \cup Y'')).$$

Then, $X \vdash \mathsf{A}(\varphi \to \neg \bigwedge(Y' \cup Y''))$ by the Modus Ponens inference rule and assumption (7). Hence, by the Distributivity axiom and the Modus Ponens, $X \vdash \mathsf{A}\varphi \to \mathsf{A}\neg \bigwedge(Y' \cup Y'')$. Thus, again by the Modus Ponens inference rule,

$$X, \mathsf{A}\varphi \vdash \mathsf{A}\neg \bigwedge(Y' \cup Y''). \tag{8}$$

At the same time, formulae $\neg \bigwedge Y' \to \neg \bigwedge(Y' \cup Y'')$ is also a tautology. Then, by the Necessitation inference rule

$$\vdash \mathsf{A}(\neg \bigwedge Y' \to \neg \bigwedge(Y' \cup Y'')).$$

Hence, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \mathsf{A}\neg \bigwedge Y' \to \mathsf{A}\neg \bigwedge(Y' \cup Y'')$. Thus, using statement (6) and the Modus Ponens inference rule, $X, \neg\mathsf{A}\varphi \vdash \mathsf{A}\neg \bigwedge(Y' \cup Y'')$. Then, $X \vdash \mathsf{A}\neg \bigwedge(Y' \cup Y'')$ by the law of excluded middle using statement (8). Hence, pair $(X, Y)$ is not consonant by Definition 11. ⊠

If the "extension" described in the previous lemma is applied *ad infinitum* for all formulae $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$, then we say that the result is a "complete" consonant pair.

**Definition 12** *A consonant pair $(X, Y)$ is called complete if, for any formula $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$, either $\neg\mathsf{A}\varphi \in X$ or $\varphi \in Y$.*

**Lemma 24** *For any consonant pair $(X, Y)$, there is a complete consonant pair $(X', Y')$ where $X \subseteq X'$ and $Y \subseteq Y'$.*

PROOF. Consider any enumeration $\varphi_1, \varphi_2, \ldots$ of all formulae in language $\Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$. For each integer $i \geq 1$ either add formula $\neg\mathsf{A}\varphi_i$ to the first set of the consonant pair or add formula $\varphi$ to the second set of the consonant pair. By Lemma 23, this could be done while maintaining the pair being consonant. Let $(X', Y')$ be the pair obtained after repeating this step for each integer $i \geq 1$. ⊠

This concludes the canonical model $M(X_0) = (S, N, A, P, I, \{\sim_a\}_{a\in A}, \pi)$ definition.

### 9.3.5 Properties of the Canonical Model

As usual, the key step in proving a completeness theorem is an "induction" or "truth" lemma. In our case, this is Lemma 29. The next four "child" lemmas are auxiliary statements used in the proof of Lemma 29. Lemma 28 is using consonant pairs.

**Lemma 25** *For any states $[\sigma]_\mathsf{S}, [\sigma']_\mathsf{S} \in P([\alpha]_\mathsf{A})$ of model $M(X_0)$ and any formula $\mathsf{D}\varphi \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$, if $[\sigma]_\mathsf{S} \sim_{[\alpha]_\mathsf{A}} [\sigma']_\mathsf{S}$ and $([\alpha]_\mathsf{A}, [\sigma']_\mathsf{S}, \varphi, [\alpha']_\mathsf{A}) \in I$, then $\varphi \in hd(\theta([\alpha']_\mathsf{A}, [\sigma']_\mathsf{S}))$.*

PROOF. Assume that $\mathsf{D}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Thus, $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})) \vdash \mathsf{KD}\varphi$ by the Introspection of Know-How axiom and the Modus Ponens rule. Hence, $\mathsf{KD}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal.

Note that $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \in [\alpha]_{\mathsf{A}}$ and $\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \in [\alpha]_{\mathsf{A}}$ by Definition 7. Then, $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \overset{\mathsf{A}}{\sim} \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$ because set $[\alpha]_{\mathsf{A}}$ is an equivalence class. Recall that $\mathsf{KD}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})$. Thus, $\mathsf{KD}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ by Lemma 17. Hence, $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})) \vdash \mathsf{D}\varphi$ by the Truth axiom and the Modus Ponens inference rule. Then, $\mathsf{D}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ is maximal. Therefore, $\varphi \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ by item 2 of Definition 9 and the assumption $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}, \varphi, [\alpha']_{\mathsf{A}}) \in I$ of the lemma. ⊠

**Lemma 26** *For any agent $[\alpha]_{\mathsf{A}} \in A$, any state $[\sigma]_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$, and any formula $\neg\mathsf{K}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$, there is $\sigma' \in \Omega$ such that $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$, $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$, and $\varphi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$.*

PROOF. Consider the set of formulae $X = \{\neg\varphi\} \cup \{\psi \mid \mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))\}$.

**Claim 3** *Set $X$ is consistent.*

PROOF OF CLAIM. Assume the opposite. Thus, there are formulae

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_n \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})) \tag{9}$$

such that $\psi_1, \dots, \psi_n \vdash \varphi$. Thus, $\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_n \vdash \mathsf{K}\varphi$ by Lemma 10. Hence, $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})) \vdash \mathsf{K}\varphi$ because of statement (9). Thus, $\neg\mathsf{K}\varphi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is consistent, which contradicts an assumption of the lemma. Therefore, set $X$ is consistent. ⊠

Let $X'$ be any maximal consistent extension of set $X$ and $\sigma'$ be sequence $\alpha :: \mathsf{K} :: X'$. The proof that $\sigma' \in \Omega$ is similar to the first part of the proof of Claim 2.

Note that $\sigma' \overset{\mathsf{A}}{\sim} \alpha$ by Definition 4. Thus, $\sigma' \in [\alpha]_{\mathsf{A}}$. Hence,

$$\sigma' \in [\alpha]_{\mathsf{A}} \cap [\sigma']_{\mathsf{S}} \tag{10}$$

since $[\sigma']_{\mathsf{S}}$ is an equivalence class. Then, $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ by Definition 6.

Note also that $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}), \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \in [\alpha]_{\mathsf{A}}$ by Definition 7. Thus, $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \overset{\mathsf{A}}{\sim} \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$. Hence, $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$ by Definition 8.

To finish the proof of the lemma, it suffices to show $\neg\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$. Indeed, by Definition 7, statement (10) implies that $\sigma' = \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$. Thus, $\neg\varphi \in X \subseteq X' = hd(\sigma') = hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$, by the choice of set $X$, the choice of set $X'$, and the choice of sequence $\sigma'$. Therefore, $\varphi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ is consistent. ⊠

**Lemma 27** *For any state $[\sigma]_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ of the canonical model and any formula $\neg\mathsf{A}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ there exists a sequence $\alpha' \in \Omega$ such that $[\sigma]_{\mathsf{S}} \in P([\alpha']_{\mathsf{A}})$ and $\varphi \notin hd(\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$.*

PROOF. Consider the set of formulae $X = \{\neg\varphi\} \cup \{\psi \mid \mathsf{A}\psi \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))\}$.

**Claim 4** *Set $X$ is consistent.*

PROOF OF CLAIM. Suppose the opposite. Thus, there are

$$\mathsf{A}\psi_1, \dots, \mathsf{A}\psi_n \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S})) \tag{11}$$

such that $\psi_1, \dots, \psi_n \vdash \varphi$. Hence, $\mathsf{A}\psi_1, \dots, \mathsf{A}\psi_n \vdash \mathsf{A}\varphi$ by Lemma 10. Then, $hd(\theta([\alpha]_\mathsf{A}, [\sigma)]_\mathsf{S}) \vdash \mathsf{A}\varphi$ by statement (11). Thus, $\neg\mathsf{A}\varphi \notin hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$ because set $hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$ is consistent, which contradicts the assumption of the lemma. Therefore, set $X$ is consistent. ⊠

Let $X'$ be any maximal consistent extension of set $X$ and $\alpha'$ be the sequence $\sigma :: \mathsf{A} :: X'$. Then, $\alpha' \overset{\mathsf{S}}{\sim} \sigma$ by Definition 4. Thus, $\alpha' \in [\sigma]_\mathsf{S}$. Hence,

$$\alpha' \in [\alpha']_\mathsf{A} \cap [\sigma]_\mathsf{S} \tag{12}$$

since $[\alpha']_\mathsf{A}$ is an equivalence class. Then, $[\sigma]_\mathsf{S} \in P([\alpha']_\mathsf{A})$ by Definition 6.

To finish the proof of the lemma, it suffices to show $\neg\varphi \in hd(\theta([\alpha']_\mathsf{A}, [\sigma]_\mathsf{S}))$. Indeed, by Definition 7, statement (12) implies that $\alpha' = \theta([\alpha']_\mathsf{A}, [\sigma]_\mathsf{S})$ . Thus, $\neg\varphi \in X \subseteq X' = hd(\alpha') = hd(\theta([\alpha']_\mathsf{A}, [\sigma]_\mathsf{S}))$, by the choice of set $X$, the choice of set $X'$, and the choice of sequence $\alpha'$. Therefore, $\varphi \notin hd(\theta([\alpha']_\mathsf{A}, [\sigma]_\mathsf{S}))$ because set $hd(\theta([\alpha']_\mathsf{A}, [\sigma]_\mathsf{S}))$ is consistent. ⊠

**Lemma 28** *For any agent $[\alpha]_\mathsf{A} \in A$, any state $[\sigma]_\mathsf{S} \in P([\alpha]_\mathsf{A})$, any formula $\neg\mathsf{D}\varphi \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$, and formula $\varphi' \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$, there are sequences $\sigma' \in \Omega$ and $\alpha' \in \Omega$ such that*

1. $[\sigma']_\mathsf{S} \in P([\alpha]_\mathsf{A})$,
2. $[\sigma]_\mathsf{S} \sim_{[\alpha]_\mathsf{A}} [\sigma']_\mathsf{S}$,
3. $([\alpha]_\mathsf{A}, [\sigma']_\mathsf{S}, \varphi', [\alpha']_\mathsf{A}) \in I$,
4. $\varphi \notin hd(\theta([\alpha']_\mathsf{A}, [\sigma']_\mathsf{S}))$.

PROOF. First, we define formula $\varphi''$ as

$$\varphi'' = \begin{cases} \varphi', & \text{if } \mathsf{D}\varphi' \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S})), \\ \top, & \text{otherwise.} \end{cases} \tag{13}$$

**Claim 5** $\mathsf{D}\varphi'' \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$.

PROOF OF CLAIM. Suppose $\mathsf{D}\varphi' \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$. Then, $\varphi'' = \varphi'$ by equation (13). Thus, $\mathsf{D}\varphi'' = \mathsf{D}\varphi' \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$.

Next, assume $\mathsf{D}\varphi' \notin hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$. Thus, $\varphi'' = \top$ by equation (13). Hence, formula $\varphi''$ is a propositional tautology. Then, $\vdash \mathsf{D}\varphi''$ by the Necessitation inference rule. Therefore, $\mathsf{D}\varphi'' \in hd(\theta([\alpha]_\mathsf{A}, [\sigma]_\mathsf{S}))$ since the set

$hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal. ⊠

Consider sets of formulae

$$X = \{\psi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}} \mid \mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))\}, Y = \{\neg\varphi, \varphi''\}.$$

Note that $\neg\mathsf{D}\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ by the assumption of the lemma and also $\mathsf{D}\varphi'' \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ by Claim 5. Hence, pair $(X, Y)$ is consonant by Lemma 22. Thus, by Lemma 24, there is a complete consonant pair $(X', Y')$ such that $X \subseteq X'$ and $Y \subseteq Y'$. Sets $X'$ and $Y'$ are consistent by Lemma 21. Let $X''$ and $Y''$ be any maximal consistent extensions of sets $X'$ and $Y'$ respectively. Define sequences

$$\sigma' = \theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) :: \mathsf{K} :: X'', \tag{14}$$
$$\alpha' = \sigma' :: \mathsf{A} :: Y''. \tag{15}$$

**Claim 6** $\sigma' \in \Omega$.

PROOF OF CLAIM. Since $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \in \Omega$, by Definition 3 and due to equation (14) it suffices to show that if $\mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$, then $\psi \in X''$. The latter follows from the choice of sets $X$, $X'$, and $X''$. ⊠

**Claim 7** $\alpha' \in \Omega$.

PROOF OF CLAIM. Since $\sigma' \in \Omega$ by Claim 6, by Definition 3 and due to equation (14) it suffices to show that if $\mathsf{A}\psi \in hd(\sigma')$, then $\psi \in Y''$. Indeed, assumption $\mathsf{A}\psi \in hd(\sigma')$ implies that $\mathsf{A}\psi \in X''$ by equation (14). Thus, $\neg\mathsf{A}\psi \notin X''$ because set $X''$ is consistent. Hence, $\neg\mathsf{A}\psi \notin X$ because $X \subseteq X' \subseteq X''$. Then, $\psi \in Y$ by Definition 12 and because $(X, Y)$ is a complete consonant pair. Therefore, $\psi \in Y \subseteq Y' \subseteq Y'' = hd(\alpha')$ by the choice of sets $Y'$ and $Y''$ and the equation (15). ⊠

**Claim 8** $\sigma' \in [\alpha]_{\mathsf{A}} \cap [\sigma']_{\mathsf{S}}$.

PROOF OF CLAIM. Note that $\sigma' \overset{\mathsf{A}}{\sim} \alpha$ by Definition 5 and because of equation (14). Thus, $\sigma' \in [\alpha]_{\mathsf{A}}$. At the same time, $\sigma' \in [\sigma']_{\mathsf{S}}$ because $[\sigma']_{\mathsf{S}}$ is an equivalence class. Therefore, $\sigma' \in [\sigma']_{\mathsf{S}} \cap [\alpha]_{\mathsf{A}}$. ⊠

**Claim 9** $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$.

PROOF OF CLAIM. The statement of the claim follows from Definition 6 and Claim 8. ⊠

**Claim 10** $\alpha' \in [\alpha']_{\mathsf{A}} \cap [\sigma']_{\mathsf{S}}$.

PROOF. Note that $\alpha' \in [\alpha']_{\mathsf{A}}$ because $[\alpha']_{\mathsf{A}}$ is an equivalence class. Also, note that $\sigma' \overset{\mathsf{S}}{\sim} \alpha'$ by Definition 4 and equation (15). Thus, $\alpha' \in [\sigma']_{\mathsf{S}}$. $\boxtimes$

**Claim 11** $[\sigma']_{\mathsf{S}} \in P([\alpha']_{\mathsf{A}})$.

PROOF OF CLAIM. The statement of the claim follows from Definition 6 and Claim 10. $\boxtimes$

**Claim 12** $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$.

PROOF OF CLAIM. Note that $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \overset{\mathsf{A}}{\sim} \sigma'$ by Definition 5 and equation (14). At the same time, $\sigma' = \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$ by Claim 8 and Definition 7. Thus, $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \overset{\mathsf{A}}{\sim} \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$. Therefore, $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$ by Definition 8, Claim 9, and Claim 11. $\boxtimes$

**Claim 13** $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}, \varphi', [\alpha']_{\mathsf{A}}) \in I$.

PROOF OF CLAIM. By Definition 9 and due to Claim 9 and Claim 11, it suffices to prove that if $\mathsf{D}\varphi' \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$, then $\varphi' \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$.

Suppose $\mathsf{D}\varphi' \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$. Note that $\sigma' = \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$ by Definition 7 and Claim 8. Hence, we have $\mathsf{D}\varphi' \in hd(\sigma')$. Thus, $hd(\sigma') \vdash \mathsf{KD}\varphi'$ by the Introspection of Knowing-Who axiom and the Modus Ponens inference rule. Then, $\mathsf{KD}\varphi' \in hd(\sigma')$ because set $hd(\sigma')$ is maximal. Hence, $\mathsf{KD}\varphi' \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ by Lemma 15 and equation (14). Thus, by the Truth axiom and the Modus Ponens inference rule $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}})) \vdash \mathsf{D}\varphi'$. Then, $\mathsf{D}\varphi' \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal. Hence, $\varphi'' = \varphi'$ by equality (13). Thus, $\varphi' \in Y \subseteq Y' \subseteq Y''$ by the choice of sets $Y$, $Y'$, and $Y''$. Then, $\varphi' \in hd(\alpha')$ by equation (15).

Finally, note that $\alpha' = \theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$ by Definition 7 and Claim 10. Therefore, $\varphi' \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$. $\boxtimes$

**Claim 14** $\varphi \notin hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$.

PROOF OF CLAIM. Note that $\neg\varphi \in Y \subseteq Y' \subseteq Y''$ by the choice of set $Y$, set $Y'$, and set $Y''$. Hence, $\neg\varphi \in hd(\alpha')$ due to equation (15). At the same time, $\alpha' = \theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}})$ by Claim 10 and Definition 7. Then, $\neg\varphi \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$. Then, $\varphi \notin hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ because set $hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ is maximal. $\boxtimes$

This concludes the proof of the lemma. $\boxtimes$

**Lemma 29** $([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \Vdash \varphi$ *iff* $\varphi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$, *for each agent* $[\alpha]_{\mathsf{A}} \in A$, *each state* $[\sigma]_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$, *and each formula* $\varphi \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$.

Proof. We prove the lemma by induction on the structural complexity of formula $\varphi$. If formula $\varphi$ is a propositional variable, then the required follows from Definition 10 and item 1 of Definition 2. If formula $\varphi$ is either a negation or an implication, then the required follows from items 2 or 3 of Definition 2 as well as the maximality and the consistency of the set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ in the standard way.

Suppose that formula $\varphi$ has the form $\mathsf{A}\psi$.

$(\Rightarrow)$ : Assume that $\mathsf{A}\psi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Thus, because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal, $\neg\mathsf{A}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Hence, by Lemma 27, there exists a sequence $\alpha' \in \Omega$ such that $[\sigma]_{\mathsf{S}} \in P([\alpha']_{\mathsf{A}})$ and $\psi \notin hd(\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Then, $([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \nVdash \psi$ by the induction hypothesis. Therefore, $([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \nVdash \mathsf{A}\psi$ by item 4 of Definition 2.

$(\Leftarrow)$ : Suppose that $\mathsf{A}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Consider any agent $[\alpha']_{\mathsf{A}} \in A$ such that $[\sigma]_{\mathsf{S}} \in P([\alpha']_{\mathsf{A}})$. By item 4 of Definition 2, it suffices to show that $([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \Vdash \psi$.

By Definition 6, the assumption $[\sigma]_{\mathsf{S}} \in P([\alpha']_{\mathsf{A}})$ of the lemma implies that $[\alpha']_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}} \neq \varnothing$. Hence, $\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \in [\alpha']_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}}$ by Definition 7. Also, $\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \in [\alpha]_{\mathsf{A}} \cap [\sigma]_{\mathsf{S}}$ by Definition 7. Thus,

$$\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \overset{\mathsf{S}}{\sim} \theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}).$$

Recall the assumption $\mathsf{A}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Then, Lemma 16 implies that $\mathsf{A}\psi \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Hence, $hd(\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}})) \vdash \psi$ by the Truth axiom. Thus, $\psi \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ because the set $hd(\theta([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal. Hence, $([\alpha']_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \Vdash \psi$ by the induction hypothesis.

Suppose that formula $\varphi$ has the form $\mathsf{K}\psi$.

$(\Rightarrow)$ : Assume that $\mathsf{K}\psi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Thus, because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$ is maximal, $\neg\mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Hence, by Lemma 26, there is a state $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ such that $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$, and $\psi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$. Then, $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \nVdash \psi$ by the induction hypothesis. Therefore, $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \nVdash \mathsf{K}\psi$ by item 5 of Definition 2.

$(\Leftarrow)$ : Assume that $\mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Consider any state $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ such that $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$. By item 5 of Definition 2, it suffices to show that $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \Vdash \psi$.

By Definition 8, the assumption $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$ implies that

$$\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \overset{\mathsf{A}}{\sim} \theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}).$$

Recall the assumption $\mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Thus, $\mathsf{K}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ by Lemma 17. Hence, $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}})) \vdash \psi$ by the Truth axiom and the Modus Ponens inference rule. Then, $\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ because set $hd(\theta([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ is maximal. Thus, $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \Vdash \psi$ by the induction hypothesis.

Finally, suppose that formula $\varphi$ has the form $\mathsf{D}\psi$.

$(\Rightarrow)$ : Assume $\mathsf{D}\psi \notin hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Consider an arbitrary name $\varphi' \in \Phi^{\mathsf{A},\mathsf{K},\mathsf{D}}$. By item 6 of Definition 2, it suffices to show that there is a state $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$

such that $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$, $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}, \varphi', [\alpha']_{\mathsf{A}}) \in I$, and $([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \nVdash \varphi$. Indeed note that, by Lemma 28, there is a state $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$ such that $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$, $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}, \varphi', [\alpha']_{\mathsf{A}}) \in I$, and $\varphi \notin hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$. Then, $([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \nVdash \varphi$ by the induction hypothesis.

$(\Leftarrow)$ : Assume that $\mathsf{D}\psi \in hd(\theta([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}))$. Thus, for any state $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$, if $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$ and $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}, \psi, [\alpha']_{\mathsf{A}}) \in I$, then $\psi \in hd(\theta([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}))$ by Lemma 25. Then, by the induction hypothesis, for any state $[\sigma']_{\mathsf{S}} \in P([\alpha]_{\mathsf{A}})$, if $[\sigma]_{\mathsf{S}} \sim_{[\alpha]_{\mathsf{A}}} [\sigma']_{\mathsf{S}}$ and $([\alpha]_{\mathsf{A}}, [\sigma']_{\mathsf{S}}, \psi, [\alpha']_{\mathsf{A}}) \in I$, then $([\alpha']_{\mathsf{A}}, [\sigma']_{\mathsf{S}}) \Vdash \psi$. Therefore, $([\alpha]_{\mathsf{A}}, [\sigma]_{\mathsf{S}}) \Vdash \mathsf{D}\psi$ by item 6 of Definition 2. $\boxtimes$

### 9.3.6 Strong Completeness

We are now ready to prove the strong completeness of our logical system.

**Theorem 5** *If $X \nvdash \varphi$, then there is an agent $a \in A$ and a state $s \in P(a)$ of a model $(S, A, P, \{\sim_a\}_{a \in A}, N, I, \pi)$ such that $(a, s) \Vdash \chi$ for each formula $\chi \in X$ and $(a, s) \nVdash \varphi$.*

Proof. Suppose that $X \nvdash \varphi$. Let $X_0$ be any maximal consistent extension of set $\{\neg\varphi\} \cup X$. Consider canonical model $M(X_0) = (S, A, P, \{\sim_a\}_{a \in A}, N, I, \pi)$. Let $\omega$ be the single-element sequence $X_0$. Thus, $\omega \in \Omega$ by Definition 3. Note that $\omega \in [\omega]_{\mathsf{A}}$ and $\omega \in [\omega]_{\mathsf{S}}$ because sets $[\omega]_{\mathsf{A}}$ and $[\omega]_{\mathsf{S}}$ are equivalence classes. Thus, $[\omega]_{\mathsf{A}} \cap [\omega]_{\mathsf{S}} \neq \varnothing$ and $\theta([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}) = \omega$. Hence, $[\omega]_{\mathsf{S}} \in P([\omega]_{\mathsf{A}})$ by Definition 6.

Note that $\chi \in X \subseteq X_0 = hd(\omega) = hd(\theta([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}))$ for each formula $\chi \in X$ and $\neg\varphi \in X_0 = hd(\omega) = hd(\theta([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}))$ by the choice of the set $X_0$, the choice of the sequence $\omega$, and equality $\theta([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}) = \omega$. Thus, $([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}) \Vdash \chi$ for each $\chi \in X$ and $([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}) \Vdash \neg\varphi$ by Lemma 29. Therefore, $([\omega]_{\mathsf{A}}, [\omega]_{\mathsf{S}}) \nVdash \varphi$ by item 2 of Definition 2. $\boxtimes$

## 10 Conclusion

In this article, we studied the properties of de dicto and de re know who. We defined both of these notions as modalities, gave their formal semantics, and have shown that neither of them is expressible through the other even if the use of two additional modalities, "knows" and "for all agents", is allowed. Finally, we gave a sound and complete logical system describing the interplay between "de dicto knows who", "knows", and "for all agents" modalities. An axiomatization of de re know who is left for future research.

# A Proof of Negative Introspection Principle

**Lemma 11** $\vdash \Box\varphi \to \Box\Box\varphi$.

PROOF. Formula $\Box\neg\Box\varphi \to \neg\Box\varphi$ is an instance of the Truth axiom. Thus, $\vdash \Box\varphi \to \neg\Box\neg\Box\varphi$ by contraposition. Hence, taking into account the following instance of the Negative Introspection axiom: $\neg\Box\neg\Box\varphi \to \Box\neg\Box\neg\Box\varphi$, we have

$$\vdash \Box\varphi \to \Box\neg\Box\neg\Box\varphi. \tag{16}$$

At the same time, $\neg\Box\varphi \to \Box\neg\Box\varphi$ is an instance of the Negative Introspection axiom. Thus, $\vdash \neg\Box\neg\Box\varphi \to \Box\varphi$ by the law of contrapositive. Hence, by the Necessitation inference rule, $\vdash \Box(\neg\Box\neg\Box\varphi \to \Box\varphi)$. Thus, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \Box\neg\Box\neg\Box\varphi \to \Box\Box\varphi$. The latter, together with statement (16), implies the statement of the lemma by propositional reasoning. $\boxtimes$

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