Knowing-How under Uncertainty

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Abstract

Logical systems containing knowledge and know-how modalities have been investigated in several recent works. Independently, epistemic modal logics in which every knowledge modality is labeled with a degree of uncertainty have been proposed. This article combines these two research lines by introducing a bimodal logic containing knowledge and know-how modalities, both labeled with a degree of uncertainty. The main technical results are soundness, completeness, and incompleteness of the proposed logical system with respect to two classes of semantics.

1. Introduction

In this article we study an interplay between knowledge, strategies, and uncertainty in multiagent systems. Consider an example of a traffic situation depicted in Figure 1, where a self-driving truck t is approaching an intersection at the same time as a regular car c. Although there is a stop sign instructing the car to yield to the truck, the car's driver does not notice the sign and does not slow down. This is detected by the radar on the self-driving truck t. The truck has two strategies that potentially can prevent a collision with the car: to accelerate or to break. How effective each of these strategies is depends on the speed of the car c. If the speed of the car is slow, the truck must accelerate to avoid being hit by the car in the rear half. If the speed is high, the truck must brake to avoid being hit in the front half.

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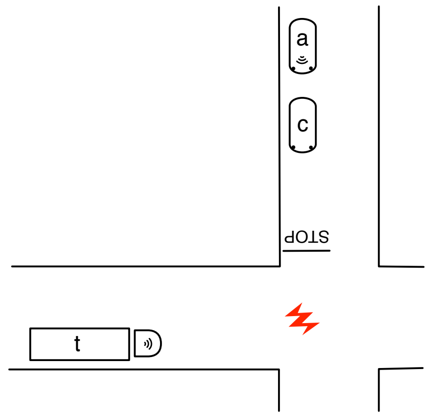

Figure 1: Road situation.

Suppose that the truck will avoid the collision by accelerating if the speed of car c is at most 58 miles per hour (mph) and that the truck will avoid the collision by breaking if the speed of the car is at least 56mph (see Figure 2). In the interval between 56 and 58mph, both strategies would allow the truck to avoid a collision.

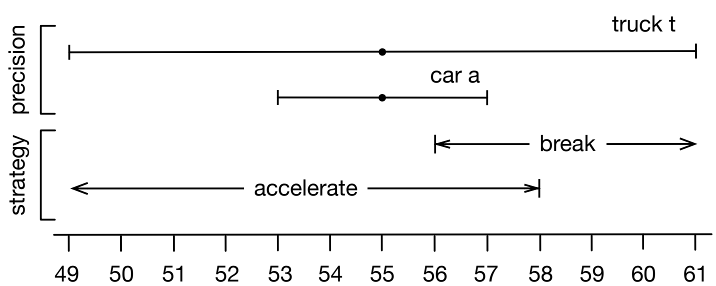

Figure 2: Strategies and precision.

Let us further assume that the actual speed of the car is 55mph, but the truck's radar can only detect the speed of the car with a precision of $\pm$6mph. Thus, the truck only knows that the speed of the car is somewhere in the interval between 49 and 61mph, see Figure 2. Thus, truck t does not know which of the two strategies would allow it to prevent a collision. Note that in this situation truck t has a strategy to avoid collision, but it does not know what this strategy is. If an agent t has a strategy to achieve goal φ, she knows that she has such a strategy, and she knows what this strategy is, then we say that she has a *know-how* strategy and denote this by $\mathsf{H}_t\varphi$. In this article we study the existence of know-how strategies to achieve a goal depending on the degree of uncertainty of the information available to the agent. We represent the degree of uncertainty by the superscript of the modality. For example, we write $\neg\mathsf{H}_t^6$(“Collision is avoided.”) to say that truck t does *not* have a know-how strategy to avoid a collision if it determines the speed of car c with a precision of $\pm$6mph. However, if the truck is able to determine the speed of the car with a precision of $\pm$2mph, then truck t has a know-how strategy to prevent the collision: H_t^2(“Collision is avoided.”).

Now suppose that an autonomous car a is driving right behind car c. From this position car a can measure the speed of car c with precision $\pm$2mph. Thus, car a knows that the speed of car c is between 53 and 57mph. Assuming that car a is aware of truck's radar precision, it can see that no matter where within the interval between 53 and 57mph the speed of car c is, truck t does *not* have a know-how strategy to avoid collision. We write this as $\mathsf{K}_a^2\neg\mathsf{H}_t^6$(“Collision is avoided.”), where modality K_a^2 denotes the knowledge of car a when it is able to determine the speed of car c with a precision of $\pm$2mph.

As another example, although statement H_t^2(“Collision is avoided.”) is true, car a does not know about this: $\neg\mathsf{K}_a^2\mathsf{H}_t^2$(“Collision is avoided.”). Indeed, due to the precision of car a's equipment, as far as car a is concerned, the speed of the car c is between 53 and 57mph. If it is 56.5mph, then statement H_t^2(“Collision is avoided.”) would not be true. A similar setting appears in many real world examples [1, 2].

The interplay between knowledge modality K_a and know-how modality H_a, both without a degree of uncertainty, has been recently studied, see Section 1.1. In this article we study the interplay between modalities K_a^c and H_a^c, where the degree of uncertainty c refers to the precision with which an agent a can position herself in an arbitrary metric space. Several “distance logics” for reasoning about modality “statement φ is true at distance at most c” were introduced in [3] without emphasizing their epistemic interpretation. We proposed the epistemic interpretation and a sound and complete system for modality K_a^c in multiagent setting [4]. The current article extends our previous work to include modality H_a^c.

Although the axiomatic system obtained in this article is a straightforward combination of existing principles, proving completeness theorems for this system required us to develop a new technique of constructing a canonical model as a tree where each child node has a twin sibling. The twin nodes are essential for the proof of Lemma 14.

1.1. Literature Review

Non-epistemic logics of coalition power were developed by Pauly [5], who also proved the completeness of the basic logic of coalition power. His approach has been widely studied in the literature [6, 7, 8, 9, 10, 11, 12, 13]. Goranko and Enqvist studied a "socially friendly" version of coalition logic in which a coalition can achieve its goal while leaving a chance to another coalition to achieve its own goal [14]. More and Naumov proposed a non-classical logical system for reasoning about a coalition achieving a goal with applications to privacy of protocols [15].

Alur, Henzinger, and Kupferman introduced Alternating-Time Temporal Logic (ATL) that combines temporal and coalition modalities [16]. Goranko and van Drimmelen gave a complete axiomatization of ATL [17]. Van der Hoek and Wooldridge proposed to combine ATL with an epistemic modality to form Alternating-Time Temporal Epistemic Logic [18]. An alternative approach to expressing the power to achieve a goal in a temporal setting is the STIT logic [19, 20, 21, 22, 23]. Broersen, Herzig, and Troquard have shown that the coalition logic can be embedded into a variation of STIT logic [24]. An alternative approach to reasoning about strategies is Strategy Logic [25, 26, 27, 28]. Unlike our current work and the works mentioned above, this logic introduces explicit quantifiers over strategies.

Know-how strategies were studied before under different names. While Jamroga and Ågotnes talked about "knowledge to identify and execute a strategy" [29], Jamroga and van der Hoek discussed "difference between an agent knowing that he has a suitable strategy and knowing the strategy itself" [30]. Van Benthem called such strategies "uniform" [31]. Broersen investigated a related notion of "knowingly doing" [32], while Broersen, Herzig, and Troquard studied the modality "know they can do" [33]. Wang captured the "knowing how" as a binary modality in a complete logical system with a single agent [34]. We previously called such strategies "executable" [35].

Since the ATL language does not contain a knowledge modality, this logic cannot distinguish between properties of strategies and know-how strategies. Instead, some works distinguish between objective (based on strategies) and subjective (based on know-how strategies) semantics of ATL [36].

Several modal logical systems that capture the interplay between knowledge and know-how strategies without uncertainty have been proposed. Ågotnes and Alechina introduced a complete axiomatization of an interplay between single-agent knowledge and coalition know-how modalities to achieve a goal in one step [37]. A modal logic that combines the distributed knowledge modality with the coalition know-how modality to maintain a goal was axiomatized by us in [35]. A sound and complete logical system in a single-agent setting for know-how strategies to achieve a goal in multiple steps rather than to maintain a goal is developed by Fervari, Herzig, Li, and Wang [38]. In [39, 40], we developed a trimodal logical system that describes an interplay between the (not know-how) coalition strategic modality, the coalition know-how modality, and the distributed knowledge modality. In [41], we proposed a logical system that combines the coalition know-how modality with the distributed knowledge modality in the perfect recall setting. In [42], we introduced a logical system for the second-order know-how. Wang proposed a complete axiomatization of "knowing how" as a binary modality [43, 34], but his logical system does not include the knowledge modality.

Uncertainty is usually formalized using probabilities [44]. Heifetz and Mongin proposed a sound and complete axiomatization of logic that uses modalities "formula φ is true with a probability *at most* p" and "formula φ is true with a probability *at least* p" [45]. Abadi and Halpern have shown that the first-order probability logic is Π^2_1-complete and, thus, does not have a finitary axiomatization [46]. Ognjanović and Rašković gave axiomatization for this logic using an infinitary inference rule [47]. Naumov and Ros gave a complete axiomatization of an extension of coalition logic with probabilities of catastrophic failures [48].

Several versions of "distance logic" were axiomatized by Kutz, Sturm, Suzuki, Wolter, and Zakharyaschev [3]. Their logical systems have modalities $\mathsf{A}^{\le c}\varphi$ and $\mathsf{A}^{>c}\varphi$ that stand for "statement φ is true at each point no further than c" and "statement φ is true everywhere at a distance more than c". In [49], Sheremet, Wolter, and Zakharyaschev introduced two new logical systems. One of them, qualitative metric logic, contains modalities $\exists^{\le c}\varphi$ (formula φ is true at some point no further than c) and $\exists^{<c}\varphi$ (formula φ is true at some point closer than c) as well as quantifiers over distances. The other system, called comparative similarity logic, is a syntactical fragment of qualitative metric logic that includes modal operators for comparing distances. They gave sound and complete axiomatisations of the second logic in several different settings.

Neither probabilistic nor distance logics discussed above include strategic modalities. In this article we combine knowledge under uncertainty modality K^c_a and strategic know-how modality H^c_a, and prove the strong completeness of the obtained system with respect to one class of semantics and the weak completeness with respect to another.

1.2. Article Outline

The rest of the article is organized as follows. In Section 2, we review the notions of metric space and finite metric space, and introduce the syntax and the semantics of our logical system. In Section 3, we list and discuss the axioms of the system. In Section 4, we give examples of formal derivations in this system. In sections that follow, we show soundness, completeness, and incompleteness of our system in different settings. We conclude in Section 9.

2. Syntax and Semantics

This section introduces the formal syntax and semantics of our logical system. Throughout the article we assume a fixed nonempty set of propositional variables and a fixed (possibly infinite) set of agents $\mathcal{A}$.

Definition 1. *Let Φ be the minimal set of formulae such that*

1. *$p \in \Phi$ for each propositional variable p,*
2. *$\neg\varphi, \varphi \to \psi \in \Phi$ for all formulae $\varphi, \psi \in \Phi$,*
3. *$\mathsf{K}_a^c\varphi, \mathsf{H}_a^c\varphi \in \Phi$ for each real number $c \geq 0$, each agent $a \in \mathcal{A}$, and each formula $\varphi \in \Phi$.*

In other words, the language Φ is defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}_a^c\varphi \mid \mathsf{H}_a^c\varphi.$$

We define Boolean constants $\top$ and $\bot$ in the usual way.

In the introductory example we assumed that the uncertainty parameter c of the modalities $\mathsf{K}_a^c\varphi$ and $\mathsf{H}_a^c\varphi$ specifies the precision with which agents know the car's position. In [4] we provided an example where an uncertainty parameter specifies the precision of a police speed radar and another example where an uncertainty parameter specifies the amount of noise in a communication channel. In the latter case, parameter c is the maximum Hamming distance between messages. Following [4], we assume that parameter c represents the precision with which the agent can determine the position (state) of *the whole system* in an arbitrary *metric space*.

In mathematics, a metric space is the most general form of the concept of distance [50]. Examples of commonly used metric spaces are Euclidean distance in $\mathbb{R}^n$, Hamming distance on strings of a fixed length, the shortest path distance on graphs, and the Manhattan distance [51] on $\mathbb{Z}^n$. In addition to these, there are Levenshtein distance [52], Damerau-Levenshtein distance [53], Jaro-Winkler distance [54, 55], and many others [56].

It is usually assumed that a distance is a non-negative real number. However, sometimes it is convenient to assume that a distance could be infinite [57], which is the approach we take in this article. In other words, we assume that the value of a distance is an extended non-negative real number, i.e., a non-negative real number or the positive infinity ∞. As usual in calculus, we assume that ∞ is greater than any real number and that the sum of ∞ and any extended non-negative real number is equal to ∞. Note, however, that per Definition 1, the formulae of our logical system can only use real numbers, not extended real numbers.

Definition 2. *A* ***metric space*** *is a pair (W, δ) such that W is a set and δ is a distance function that maps every pair of elements of W to an extended non-negative real number, where the following properties hold for all $u, v, w \in W$:*

1. *Identity of Indiscernibles: $\delta(u, v) = 0$ iff $u = v$,*
2. *Symmetry: $\delta(u, v) = \delta(v, u)$,*
3. *Triangle Inequality: $\delta(u, v) \leq \delta(u, w) + \delta(w, v)$.*

Definition 3. *A metric space (W, δ) is* ***finite*** *if all values of distance function δ are real numbers.*

The next definition specifies the class of models for our logical system. By X^Y we denote the set of all functions from set Y to set X.

Definition 4. *An* ***epistemic transition system*** *is a tuple $(W, \{\delta_a\}_{a \in \mathcal{A}}, D, M, \pi)$, where*

1. *W is a set of "epistemic states",*
2. *(W, δ_a) is a metric space for each agent $a \in \mathcal{A}$,*
3. *D is a nonempty set called "domain of actions",*
4. *$M \subseteq W \times D^{\mathcal{A}} \times W$ is a "transition mechanism",*
5. *π maps propositional variables to subsets of W.*

Informally, a model of our logical system consists of a set of states with agent-specific metrics. It resembles an S5 Kripke model except that, instead of having an indistinguishability relation specific to each agent, the model has a metric specific to an agent. If distance $\delta_a(u, v)$ is equal to infinity, then agent a can *always* distinguish epistemic states u and v. The assumption that agents have agent-specific metrics is natural in the setting when the agents can measure different sets of parameters of the system.

In each state, agents take actions. The set of all actions taken, called an *action profile*, is viewed as a function from the set of all agents $\mathcal{A}$ to a "domain of actions" D. In other words, an action profile is an element of set $D^{\mathcal{A}}$. Although this was not emphasized in our introductory example, we assume that once the actions are taken, the system transitions from one state to another. Thus, we call the model an *epistemic transition system*. The rules that determine the next state based on the current state and the action profile are captured by a *transition mechanism* M. Note that these rules are, generally speaking, non-deterministic. Furthermore we assume that in some situations there might be no "next" state(s). We interpret this as a termination of the transition system.

Definition 5. *An epistemic transition system* $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$ *is* ***with finite metrics*** *if* (W, δ_a) *is a finite metric space for each agent* $a \in \mathcal{A}$*.*

In this article we prove that our logical system is strongly complete with respect to all epistemic transition systems (Theorem 1) and weakly complete with respect to all epistemic transition systems with finite metrics (Theorem 2).

The next definition is the key definition of this section. It formally specifies the meaning of modalities K^c_a and H^c_a. The part pertaining to modality K^c_a is identical to the corresponding definition in [4].

Definition 6. *For any epistemic state* $w \in W$ *of an epistemic transition system* $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$ *and any formula* $\varphi \in \Phi$*, let the* ***satisfiability relation*** $w \Vdash \varphi$ *be defined recursively as follows:*

1. $w \Vdash p$ *if* $w \in \pi(p)$*, where* p *is a propositional variable,*
2. $w \Vdash \neg\varphi$ *if* $w \nVdash \varphi$*,*
3. $w \Vdash \varphi \to \psi$ *if* $w \nVdash \varphi$ *or* $w \Vdash \psi$*,*
4. $w \Vdash \mathsf{K}^c_a\varphi$ *if* $w' \Vdash \varphi$ *for each epistemic state* $w' \in W$ *such that* $\delta_a(w, w') \leq c$*,*

5. *$w \Vdash \mathsf{H}_a^c\varphi$ if there is an action $\alpha \in D$ such that $w'' \Vdash \varphi$ for all epistemic states $w', w'' \in W$ and each action profile $s \in D^{\mathcal{A}}$ where $\delta_a(w, w') \leq c$, $s(a) = \alpha$, and $(w', s, w'') \in M$, see Figure 3.*

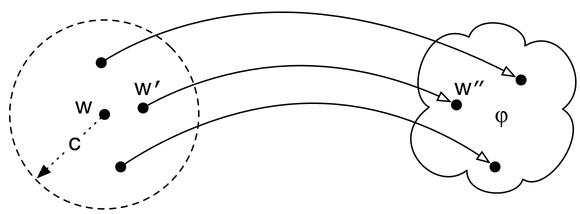

Figure 3: Towards the definition of the satisfiability relation for modality $\mathsf{H}_a^c\varphi$.

In other words, $w \Vdash \mathsf{K}_a^c\varphi$ if formula φ is satisfied at each point (state) in a ball of radius c around point w defined by the metric δ_a. Also, $w \Vdash \mathsf{H}_a^c\varphi$ if there is an action of agent a that achieves goal φ from any point in the ball described above. In the case when $c = 0$, formulae $w \Vdash \mathsf{K}_a^c\varphi$ and $w \Vdash \mathsf{H}_a^c\varphi$ have special meanings. The first of them states that φ is true just at point w. Thus, formula $\mathsf{K}_a^0\varphi$ and formula φ are logically equivalent. This fact is captured through the combination of the Zero Confidence and the Truth axioms of our logical system that we introduce in the next section. Similarly, formula $\mathsf{H}_a^0\varphi$ states that a strategy to achieve φ exists at point w.

Epistemic transition systems are similar to the semantics of Coalition Logic [58, 5] and concurrent game structures, the semantics of ATL [16], with three notable differences. First, in those semantics, the domain of choices depends on a state and an agent. On the other hand, we assume a uniform domain of choices for all states and all agents. This difference is insignificant because multiple domains of choices could be replaced with their union if the aggregation mechanism is modified to interpret the additional choices as alternative names for the original choices. Second, unlike the transition *function* in these semantics, our aggregation mechanism allows to capture nondeterministic transitions. This difference is significant because restricting semantics to only deterministic transitions would require additional axioms. For example, property $\mathsf{H}_a^0\varphi \vee \mathsf{H}_a^0\neg\varphi$ is universally true in single-agent deterministic transition systems, but is not universally true in single-agent nondeterministic systems. Third, we do *not* require that, for any current state and any action profile, there is at least one next state. Thus, in our setting, the system may terminate. Hence, for example, formula $\mathsf{H}_a^c\bot$ might be satisfied in some states of our epistemic transition systems.

3. Axioms

In addition to propositional tautologies in language Φ, our logical system has the following five axioms:

1. Zero Confidence: $\varphi \to \mathsf{K}_a^0\varphi$,
2. Truth: $\mathsf{K}_a^c\varphi \to \varphi$,
3. Negative Introspection: $\neg\mathsf{K}_a^c\varphi \to \mathsf{K}_a^d\neg\mathsf{K}_a^{c+d}\varphi$,
4. Distributivity: $\mathsf{K}_a^c(\varphi \to \psi) \to (\mathsf{K}_a^c\varphi \to \mathsf{K}_a^c\psi)$,
5. Strategic Positive Introspection: $\mathsf{H}_a^{c+d}\varphi \to \mathsf{K}_a^c\mathsf{H}_a^d\varphi$.

The first four of these axioms come from [4]. The Strategic Positive Introspection axiom without a degree of uncertainty first appeared in [37] and is also present in [34, 38, 39, 35, 41, 42, 40]. Blending the know-how and the degree of uncertainty lines of research into one logical system that captures non-trivial interplay between the two notions is the main contribution of this article.

We write $\vdash \varphi$ if formula $\varphi \in \Phi$ is provable from the above axioms using the Monotonicity, K-Necessitation, H-Necessitation, and Modus Ponens inference rules:

$$\frac{\varphi \to \psi}{\mathsf{H}_a^c\varphi \to \mathsf{H}_a^c\psi}, \qquad \frac{\varphi}{\mathsf{K}_a^c\varphi}, \qquad \frac{\varphi}{\mathsf{H}_a^c\varphi}, \qquad \frac{\varphi, \varphi \to \psi}{\psi}.$$

If $\vdash \varphi$, then we say that statement φ is a *theorem* of our logical system.

We write $X \vdash \varphi$ if formula $\varphi \in \Phi$ is provable from the theorems of our logical system and an additional set of axioms X using only the Modus Ponens inference rule. Note that if set X is empty, then statement $X \vdash \varphi$ is equivalent to $\vdash \varphi$. We say that set X is *consistent* if $X \nvdash \bot$.

Lemma 1 (deduction). *If $X, \varphi \vdash \psi$, then $X \vdash \varphi \to \psi$.*

Proof. Suppose that sequence $\psi_1, \dots, \psi_n$ is a proof from set $X \cup \{\varphi\}$ and the theorems of our logical system that uses the Modus Ponens inference rule only. In other words, for each $k \le n$, either

1. $\vdash \psi_k$, or
2. $\psi_k \in X$, or
3. ψ_k is equal to φ, or
4. there are $i, j < k$ such that formula ψ_j is equal to $\psi_i \to \psi_k$.

It suffices to show that $X \vdash \varphi \to \psi_k$ for each $k \leq n$. We prove this by induction on k through considering the four cases above separately.

Case 1: $\vdash \psi_k$. Note that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Hence, $\vdash \varphi \to \psi_k$ by the Modus Ponens inference rule. Therefore, $X \vdash \varphi \to \psi_k$.

Case 2: $\psi_k \in X$. Then, $X \vdash \psi_k$.

Case 3: formula ψ_k is equal to φ. Thus, $\varphi \to \psi_k$ is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$.

Case 4: formula ψ_j is equal to $\psi_i \to \psi_k$ for some $i, j < k$. Thus, by the induction hypothesis, $X \vdash \varphi \to \psi_i$ and $X \vdash \varphi \to (\psi_i \to \psi_k)$. Note that formula $(\varphi \to \psi_i) \to ((\varphi \to (\psi_i \to \psi_k)) \to (\varphi \to \psi_k))$ is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$ by applying the Modus Ponens inference rule twice. $\boxtimes$

Note that it is important for the above proof that $X \vdash \varphi$ stands for derivability only using the Modus Ponens inference rule. For example, if K-Necessitation is allowed, then the proof will have to include one more case where ψ_k is formula $\mathsf{K}^c_a\psi_i$ for some real number $c \geq 0$, some agent $a \in \mathcal{A}$, and some integer $i < k$. In this case we will need to prove that if $X \vdash \varphi \to \psi_i$, then $X \vdash \varphi \to \mathsf{K}^c_a\psi_i$, which is not true.

Lemma 2 (Lindenbaum). *Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

Proof. The standard proof of Lindenbaum's lemma applies here [59, Proposition 2.14]. However, since the formulae in our logical system use real numbers in superscript, the set of formulae is uncountable. Thus, the proof of Lindenbaum's lemma in our case relies on the Axiom of Choice. $\boxtimes$

4. Examples of Derivations

In this section we give two examples of formal derivations in our logical system. Both of these results are used later in the proof of Theorem 1. The first example shows the monotonicity of modality H with respect to the degree of uncertainty.

Lemma 3. $\vdash \mathsf{H}^c_a\varphi \to \mathsf{H}^d_a\varphi$*, where* $d \leq c$.

Proof. By the Strategic Positive Introspection axiom, assumption $d \leq c$ implies that $\vdash \mathsf{H}_a^c\varphi \to \mathsf{K}_a^{c-d}\mathsf{H}_a^d\varphi$. At the same time, by the Truth axiom, $\vdash \mathsf{K}_a^{c-d}\mathsf{H}_a^d\varphi \to \mathsf{H}_a^d\varphi$. Therefore, by the laws of propositional reasoning, $\vdash \mathsf{H}_a^c\varphi \to \mathsf{H}_a^d\varphi$. ☒

In [4], the Positive Introspection principle for modality K_a^c is an additional axiom. Next we show that, just like in the case of the logic S5, this principle is in fact derivable from the axioms for modality K_a^c listed in the previous section.

Lemma 4. $\vdash \mathsf{K}_a^{c+d}\varphi \to \mathsf{K}_a^c\mathsf{K}_a^d\varphi$.

Proof. Note that formula $\mathsf{K}_a^0\neg\mathsf{K}_a^{c+d}\varphi \to \neg\mathsf{K}_a^{c+d}\varphi$ is an instance of the Truth axiom. Thus, by contraposition,

$$\vdash \mathsf{K}_a^{c+d}\varphi \to \neg\mathsf{K}_a^0\neg\mathsf{K}_a^{c+d}\varphi \tag{1}$$

Also,

$$\neg\mathsf{K}_a^0\neg\mathsf{K}_a^{c+d}\varphi \to \mathsf{K}_a^c\neg\mathsf{K}_a^{0+c}\neg\mathsf{K}_a^{c+d}\varphi \tag{2}$$

is an instance of the Negative Introspection axiom. Additionally, formula $\neg\mathsf{K}_a^d\varphi \to \mathsf{K}_a^c\neg\mathsf{K}_a^{c+d}\varphi$ is an instance of the Negative Introspection axiom. Thus, $\vdash \neg\mathsf{K}_a^c\neg\mathsf{K}_a^{c+d}\varphi \to \mathsf{K}_a^d\varphi$ by the law of contrapositive in propositional logic. Hence, $\vdash \mathsf{K}_a^c(\neg\mathsf{K}_a^c\neg\mathsf{K}_a^{c+d}\varphi \to \mathsf{K}_a^d\varphi)$ by the Necessitation inference rule. Thus, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{K}_a^c\neg\mathsf{K}_a^c\neg\mathsf{K}_a^{c+d}\varphi \to \mathsf{K}_a^c\mathsf{K}_a^d\varphi. \tag{3}$$

By the laws of propositional reasoning, statements (1), (2), and (3), imply the statement of the lemma. ☒

5. Soundness

The soundness of the inference rules is straightforward. Below we show the soundness of the Zero Confidence, the Truth, the Negative Introspection, the Distributivity, and the Strategic Positive Introspection axioms as separate lemmas. In these lemmas, w is an arbitrary state of an epistemic transition system, $\varphi, \psi \in \Phi$ are formulae, $a \in \mathcal{A}$ is an agent, and $c, d \geq 0$ are real numbers.

Lemma 5 (Zero Confidence). *If $w \Vdash \varphi$, then $w \Vdash \mathsf{K}_a^0\varphi$.*

Proof. We need to prove that $u \Vdash \varphi$ for each epistemic state $u \in W$ such that $\delta_a(w, u) = 0$. By the Identity of Indiscernibles property of metric spaces (see Definition 2), equality $\delta_a(w, u) = 0$ implies $w = u$. Therefore, $u \Vdash \varphi$ by the assumption $w \Vdash \varphi$. $\boxtimes$

Lemma 6 (Truth). *If $w \Vdash \mathsf{K}_a^c\varphi$, then $w \Vdash \varphi$.*

Proof. Suppose that $w \Vdash \mathsf{K}_a^c\varphi$. By the Identity of Indiscernibles property of metric spaces (see Definition 2), $\delta_a(w, w) = 0$. Thus, $\delta_a(w, w) \leq c$. Therefore, $w \Vdash \varphi$ by Definition 6. $\boxtimes$

Lemma 7 (Negative Introspection). *If $w \Vdash \neg\mathsf{K}_a^c\varphi$, then $w \Vdash \mathsf{K}_a^d\neg\mathsf{K}_a^{c+d}\varphi$.*

Proof. By Definition 6, assumption $w \Vdash \neg\mathsf{K}_a^c\varphi$ implies the existence of an epistemic state $u \in W$ such that $\delta_a(w, u) \leq c$ and $u \nVdash \varphi$. To prove $w \Vdash \mathsf{K}_a^d\neg\mathsf{K}_a^{c+d}\varphi$, consider any epistemic state $v \in W$ such that $\delta_a(w, v) \leq d$. It suffices to show that $v \nVdash \mathsf{K}_a^{c+d}\varphi$. Indeed, by the Triangle Inequality, $\delta_a(v, u) \leq \delta_a(v, w) + \delta_a(w, u) \leq d + c$. Additionally, $u \nVdash \varphi$ due to the choice of state u. Therefore, by Definition 6, $v \nVdash \mathsf{K}_a^{c+d}\varphi$. $\boxtimes$

Lemma 8 (Distributivity). *If $w \Vdash \mathsf{K}_a^c(\varphi \to \psi)$ and $w \Vdash \mathsf{K}_a^c\varphi$, then $w \Vdash \mathsf{K}_a^c\psi$.*

Proof. Consider any epistemic state $u \in W$ such that $\delta_a(w, u) \leq c$. It suffices to show that $u \Vdash \psi$. Indeed, $u \Vdash \varphi \to \psi$ and $u \Vdash \varphi$ by Definition 6 item 4 and assumptions $w \Vdash \mathsf{K}_a^c(\varphi \to \psi)$ and $w \Vdash \mathsf{K}_a^c\varphi$ respectively. Therefore, $u \Vdash \psi$ by item 3 of Definition 6. $\boxtimes$

Lemma 9 (Strategic Positive Introspection). *If $w \Vdash \mathsf{H}_a^{c+d}\varphi$, then $w \Vdash \mathsf{K}_a^c\mathsf{H}_a^d\varphi$.*

Proof. By Definition 6, assumption $w \Vdash \mathsf{H}_a^{c+d}\varphi$ implies that there is an action $\alpha \in D$ such that for all states $w', w'' \in W$ and each action profile $s \in D^{\mathcal{A}}$, if $\delta(w, w') \le c + d$, $s(a) = \alpha$, and $(w', s, w'') \in M$, then $w'' \Vdash \varphi$.

Consider any state $u \in W$ such that $\delta_a(w, u) \le c$. By Definition 6, it suffices to show that $u \Vdash \mathsf{H}_a^d\varphi$. Towards this goal, consider arbitrary epistemic states $u', u'' \in W$ and an arbitrary action profile $s \in D^{\mathcal{A}}$ such that $\delta_a(u, u') \le d$, $s(a) = \alpha$, and $(u', s, u'') \in M$. By Definition 6, it suffices to show that $u'' \Vdash \varphi$. Indeed, by the Triangle Inequality,

$$\delta_a(w, u') \le \delta_a(w, u) + \delta_a(u, u') \le c + d.$$

Therefore, $u'' \Vdash \varphi$ by the choice of action α. ⊠

6. Strong Completeness

In this section we prove the strong completeness of our logical system with respect to the class of all epistemic transition systems. We start by defining a canonical epistemic transition system $(W, \{\delta_a\}_{a \in \mathcal{A}}, D, M, \pi)$. The states of a canonical model are often defined as maximal consistent sets of formulae and the rest of the model is defined through these states. This approach does not appear to work in our case. Indeed, by Definition 6, for any two epistemic states $w, u \in W$ and any formula $\mathsf{K}_a^c\varphi$, we want to have the following property: if $\mathsf{K}_a^c\varphi \in w$ and $\neg\varphi \in u$, then $\delta_a(w, u) > c$. Thus, it would be natural to define metric δ_a as

$$\delta_a(w, u) = \inf\{c \ge 0 \mid \text{ there is } \mathsf{K}_a^c\varphi \in w \text{ such that } \neg\varphi \in u\}.$$

The problem with this definition is that the infimum might belong to the set. If this is the case, there exists a formula $\mathsf{K}_a^c\varphi \in w$ such that $\neg\varphi \in u$ and $c = \delta_a(w, u)$, which is inconsistent with our original intention. A possible way around this issue is to define δ_a as

$$\delta_a(w, u) = \inf\{c \ge 0 \mid \text{ there is } \mathsf{K}_a^c\varphi \in w \text{ such that } \neg\varphi \in u\} - \varepsilon$$

for some number ε. Of course, ε should be sufficiently small because we also want to have another property: if $\delta_a(w, u) \le c$ and $\mathsf{K}_a^c\varphi \in w$, then $\varphi \in u$. Sufficiently small ε can be chosen if set w is finite, but not when it is infinite. For example, it can be used when w is a maximal consistent set of subformulae of some formula. Thus, this approach could be used to prove weak completeness, but not strong completeness. A version of this approach is used in [3] to prove the weak completeness for a logical system without the know-how modality.

Since our goal is to prove the strong completeness theorem, in this section we use a different approach. Instead of defining metric between two maximal consistent sets based on the sets themselves, we "superimpose" a metric on the sets. Namely, we define a forest (a set of trees) whose nodes are maximal consistent sets and whose edges are labeled by positive real numbers representing their lengths. The nodes will play a role of states and the distance between two states is defined as the length of a simple path connecting the states. It is interesting to point out that in [40] we also used a tree (but not a forest) to construct canonical models. However, the reason for a tree in [40] is the distributed knowledge, not a metric.

We are now ready to describe the tree construction. It represents epistemic states as sequences of maximal consistent sets with some additional data sandwiched in between. The data consists of an agent, a degree of uncertainty, and a Boolean value. Having a Boolean value is an original contribution of this article. It is also probably the most interesting part of the canonical model construction. It allows each node (except for the root nodes) to have a twin sibling node. In Lemma 14 we observe that if a node is not far from each of the twin nodes, then it is not far from the parent node. The contrapositive of this lemma, if a node is far from the parent of the twins, then it is far from at least one of the twins, is used in the proof of Lemma 19.

Definition 7. *The set of epistemic states W consists of all sequences*

$$X_0, (a_1, c_1, \ell_1), X_1, \dots, (a_n, c_n, \ell_n), X_n$$

such that

1. $n \geq 0$,
2. X_i *is a maximal consistent subset of* Φ *for each* $i \geq 0$,
3. $a_i \in \mathcal{A}$ *for each* $i \geq 1$,
4. c_i *is a positive real number for each* $i \geq 1$,
5. $\ell_i \in \{0, 1\}$ *is a Boolean value for each* $i \geq 1$,
6. *if* $\mathsf{K}_{a_i}^{c_i}\varphi \in X_{i-1}$, *then* $\varphi \in X_i$ *for each* $i \geq 1$ *and each formula* $\varphi \in \Phi$.

For any sequence $s = x_1, x_2, \dots, x_n$ and any element y, by $hd(s)$ we mean the element x_n and by $s :: y$ we mean the sequence $x_1, x_2, \dots, x_n, y$.

Lemma 10. *For any epistemic state $w :: (a, c, \ell) :: X \in W$ and any real number $d \geq c$,*

1. *if $\mathsf{K}_a^d\varphi \in hd(w)$, then $\mathsf{K}_a^{d-c}\varphi \in X$,*
2. *if $\mathsf{K}_a^d\varphi \in X$, then $\mathsf{K}_a^{d-c}\varphi \in hd(w)$.*

Proof. To prove the first statement, suppose that $\mathsf{K}_a^d\varphi \in hd(w)$. Thus, $hd(w) \vdash \mathsf{K}_a^c\mathsf{K}_a^{d-c}\varphi$ by Lemma 4, the Modus Ponens inference rule, and the assumption $d \geq c$. Hence, $\mathsf{K}_a^c\mathsf{K}_a^{d-c}\varphi \in hd(w)$ by the maximality of the set $hd(w)$. Therefore, $\mathsf{K}_a^{d-c}\varphi \in X$ by Definition 7.

To prove the second statement, suppose that $\mathsf{K}_a^{d-c}\varphi \notin hd(w)$. Hence, $\neg\mathsf{K}_a^{d-c}\varphi \in hd(w)$ by the maximality of set $hd(w)$. Thus, $hd(w) \vdash \mathsf{K}_a^c\neg\mathsf{K}_a^d\varphi$ by the Negative Introspection axiom. Hence, $\mathsf{K}_a^c\neg\mathsf{K}_a^d\varphi \in hd(w)$ due to the maximality of set $hd(w)$. Then, $\neg\mathsf{K}_a^d\varphi \in X$ by Definition 7, which contradicts the assumption $\mathsf{K}_a^d\varphi \in X$ and the consistency of set X. $\boxtimes$

We say that epistemic states $w_1, w_2 \in W$ are *adjacent* if one of them is obtained from the other by removing the last two elements of the sequence. For example, epistemic states $X_0, (a, 1.3, 1), X_1$ and $X_0, (a, 1.3, 1), X_1, (b, 0.5, 0), X_2$ are adjacent. Although any node w of the graph is a sequence, it is convenient to visualize this graph by labeling node w with $hd(w)$ and labeling each edge with a triple of the form (a, c, ℓ). For example, in Figure 4, the edge between nodes $X_0, (a, 1.3, 1), X_1$ and $X_0, (a, 1.3, 1), X_1, (b, 0.5, 0), X_2$ is labeled with triple $(b, 0.5, 0)$.

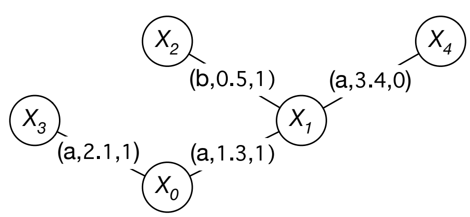

Figure 4: A fragment of the forest formed by sequences.

By a *simple path* we mean any sequence of distinct nodes $w_1, \dots, w_n$ such that $n \geq 1$ and nodes w_i and w_{i+1} are adjacent for each $i < n$.

Lemma 11. *The adjacency relation on set W forms a graph without cycles.*

Proof. Suppose that the graph has a simple cycle of length at least 3. Consider a node w on this cycle whose length (the number of elements in the sequence) is the largest. Let u and v be the adjacent nodes to w on the simple cycle. By the above definition of the adjacency, any two adjacent nodes have different lengths. Thus, since node w has the largest length, nodes u and v must have shorter lengths (as sequences) than node w. Hence, again by the definition of the adjacency, $u = v$. Thus, the simple cycle has length 2, which is a contradiction. ⊠

Definition 8. *If all edges along the simple path between nodes w and w' are labeled with triples whose first component is agent a, then $\delta_a(w, w')$ is the sum of all second components of the labels along this path. Otherwise, $\delta_a(w, w') = \infty$.*

For the example depicted in Figure 4,

$$\begin{aligned}
&\delta_a((X_0, (a, 2.1, 1), X_3), (X_0, (a, 1.3, 1), X_1, (a, 3.4, 0), X_4)) \\
&\qquad = 2.1 + 1.3 + 3.4 = 6.8, \\
&\delta_a((X_0, (a, 2.1, 1), X_3), (X_0, (a, 1.3, 1), X_1, (b, 0.5, 0), X_2)) \\
&\qquad = \infty.
\end{aligned}$$

Lemma 12. *(W, δ_a) is a metric space for each agent $a \in \mathcal{A}$.*

Proof. Consider any states $w, u, v \in W$. By Definition 2, it suffices to show that the triangle inequality $\delta_a(w, v) \leq \delta_a(w, u) + \delta_a(u, v)$ holds. First, note that if either $\delta_a(w, u) = \infty$ or $\delta_a(u, v) = \infty$, then the triangle inequality is true.

Suppose now that $\delta_a(w, u)$ and $\delta_a(u, v)$ are real numbers. Thus, there is a simple path of length $\delta_a(w, u)$ from node w to node u and a simple path of length $\delta_a(u, v)$ from node u to node v such that all edges along both paths are labeled with triples whose first component is a. Hence, there is a path of length $\delta_a(w, u) + \delta_a(u, v)$ from node w to node v such that all edges along this path are labeled with triples whose first component is a. Therefore, the length $\delta_a(w, v)$ of a *simple* path from w to v such that all edges along this path are labeled with triples whose first component is a is at most $\delta_a(w, u) + \delta_a(u, v)$. ⊠

Lemma 13. *For any epistemic state w, any agent $a \in \mathcal{A}$, and any real number $c > 0$, both sequence $w :: (a, c, 0) :: hd(w)$ and sequence $w :: (a, c, 1) :: hd(w)$ are epistemic states.*

Proof. Let $w' = w :: (a, c, 0) :: hd(w)$. Note that $hd(w') = hd(w)$. Thus, by Definition 7, to show that $w' \in W$, it suffices to show that if $\mathsf{K}_a^c\varphi \in hd(w)$, then $\varphi \in hd(w') = hd(w)$. The latter follows from the Truth axiom and the maximality of set $hd(w)$. The case when $w' = w :: (a, c, 1) :: hd(w)$ is similar. ⊠

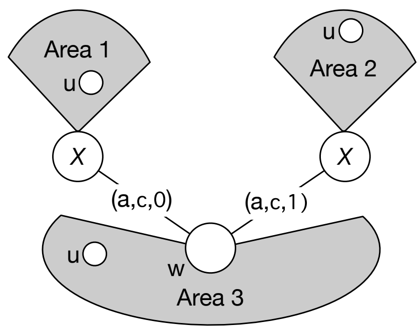

Figure 5: Towards the proof of Lemma 14.

By Definition 7, sequence $w :: (a, c, 0) :: X$ is an epistemic state if and only if sequence $w :: (a, c, 1) :: X$ is also an epistemic state. We informally refer to these states as twin children of node w. Twin children are crucial for our construction of the canonical transition system. The need for such children will become clear in the proof of Lemma 19. The next lemma shows a fundamental property of the twin children: if a node is not far from each of the twin nodes, then it is not far from the parent node.

Lemma 14. *For any real number d, any epistemic state $u \in W$, any agent $a \in \mathcal{A}$, and any two epistemic states $w :: (a, c, 0) :: X \in W$ and $w :: (a, c, 1) :: X \in W$, if $\delta_a(u, w :: (a, c, 0) :: X) \leq d$ and $\delta_a(u, w :: (a, c, 1) :: X) \leq d$, then $\delta_a(u, w) \leq d - c$.*

Proof. Since number d is finite (not ∞), by Definition 8, the assumption $\delta_a(u, w :: (a, c, 0) :: X) \leq d$ and the assumption $\delta_a(u, w :: (a, c, 1) :: X) \leq d$ imply that node u is located in the same tree of the forest as node w, node $w :: (a, c, 0) :: X$ and node $w :: (a, c, 1) :: X$. Hence, node u is located either in the subtree of node $w :: (a,c,0) :: X$, or in the subtree of node $w :: (a,c,1) :: X$, or in the tree, but outside of these subtrees. We refer to these three locations as Area 1, Area 2, and Area 3, see Figure 5. If node u is located in Area 1 or Area 3, then the path from node u to node $w :: (a,c,1) :: X$ goes through node w. Thus, assumption $\delta_a(u, w :: (a,c,1) :: X) \leq d$ implies that $\delta_a(u,w) \leq d-c$. If node u is located in Area 2, then the path from node u to node $w :: (a,c,0) :: X$ goes through node w. Thus, assumption $\delta_a(u, w :: (a,c,0) :: X) \leq d$ implies that $\delta_a(u,w) \leq d-c$. ⊠

Lemma 15. *For any epistemic states $w, w' \in W$, any agent $a \in \mathcal{A}$, any real number $c \geq 0$, and any formula $\varphi \in \Phi$, if $\delta_a(w,w')$ is finite and $\mathsf{K}_a^{c+\delta_a(w,w')}\varphi \in hd(w)$, then $\mathsf{K}_a^c\varphi \in hd(w')$.*

Proof. The statement of the lemma is proven by the induction on the length of the path between nodes w and w' using Lemma 10 and Definition 8. ⊠

Next, we define the domain D of actions and the mechanism M of the canonical epistemic transition system $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$. Informally, if $\mathsf{H}_a^c\varphi \in hd(u)$, then agent a has an action that can be used in state u to achieve φ in the next state. Such an action is represented by a triple (φ, u, c).

Definition 9. *The domain of actions D consists of all triples (φ, u, c) such that $\varphi \in \Phi$, $u \in W$, and $c \geq 0$.*

Furthermore, to match Definition 6, agent a should be able to use the same action (φ, u, c) at any state w' in a ball of radius c around state u, see Figure 6. This condition is explicitly enforced in the definition of the

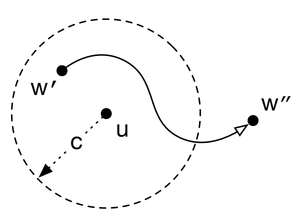

Figure 6: Canonical mechanism.

canonical mechanism M given below.

Definition 10. *Let the mechanism M be the set of all triples $(w', s, w'') \in W \times D^{\mathcal{A}} \times W$ such that for each agent $a \in \mathcal{A}$, each formula $\varphi \in \Phi$, each state $u \in W$, and each real number $c \geq 0$, if $\mathsf{H}_a^c\varphi \in hd(u)$, $s(a) = (\varphi, u, c)$, and $\delta_a(w', u) \leq c$, then $\varphi \in hd(w'')$.*

Definition 11. $\pi(p) = \{w \in W \mid p \in hd(w)\}$.

This concludes the definition of the canonical epistemic transition system $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$. The next five technical lemmas establish basic properties of the epistemic states in the system. They are used in the proof of Lemma 21.

Note that per Lemma 15, a formula of the form $\mathsf{K}_a^c\varphi$ is propagated along each path whose edges are labeled with agent a. At the same time, as the proof of the next lemma shows, a witness state for a formula of the form $\neg\mathsf{K}_a^c\varphi$ can be constructed in just one step.

Lemma 16. *For any state $w \in W$ and any formula $\neg\mathsf{K}_a^c\varphi \in hd(w)$, there is a state $w' \in W$ such that $\delta_a(w, w') \leq c$ and $\neg\varphi \in hd(w')$.*

Proof. First, suppose that $c = 0$. Thus, $hd(w) \vdash \neg\varphi$ by the contrapositive of the Zero Confidence axiom and the assumption $\neg\mathsf{K}_a^c\varphi \in hd(w)$ of the lemma. Hence, $\neg\varphi \in hd(w)$ due to the maximality of the set $hd(w)$. Note that $\delta_a(w, w) = 0 \leq c$ by the Identity of Indiscernibles from Definition 2. Choose w' to be w.

Next assume that $c > 0$. Consider set $Y = \{\neg\varphi\} \cup \{\psi \mid \mathsf{K}_a^c\psi \in hd(w)\}$. Let us first show that this set is consistent. Suppose the opposite. Thus, there are formulae $\mathsf{K}_a^c\psi_1, \dots, \mathsf{K}_a^c\psi_n \in hd(w)$ such that $\psi_1, \dots, \psi_n \vdash \varphi$. Then, by Lemma 1 applied n times,

$$\vdash \psi_1 \to (\psi_2 \to \dots (\psi_n \to \varphi) \dots).$$

Hence, by the K-Necessitation inference rule,

$$\vdash \mathsf{K}_a^c(\psi_1 \to (\psi_2 \to \dots (\psi_n \to \varphi) \dots).$$

Thus, by the Distributivity axiom and the Modus Ponens rule,

$$\vdash \mathsf{K}_a^c\psi_1 \to \mathsf{K}_a^c(\psi_2 \to \dots (\psi_n \to \varphi) \dots).$$

Then, due to the assumption $\mathsf{K}_a^c\psi_1 \in hd(w)$ and the Modus Ponens inference rule, $hd(w) \vdash \mathsf{K}_a^c(\psi_2 \to \dots(\psi_n \to \varphi)\dots)$. By repeating the previous step $n-1$ times, we have $hd(w) \vdash \mathsf{K}_a^c\varphi$, which contradicts the assumption $\neg\mathsf{K}_a^c\varphi \in hd(w)$ and the consistency of the set $hd(w)$. Therefore, set Y is consistent. By Lemma 2, there is a maximal consistent extension Y' of set Y. Let w' be sequence $w :: (a, c, 0) :: Y'$. Note that $w' \in W$ by Definition 7. Also, $\delta_a(w, w') = c$ by Definition 8. Finally, $\neg\varphi \in Y \subseteq Y' = hd(w')$ by the choice of set Y, set Y', and sequence w'. ⊠

Lemma 17. *For all epistemic states $w, w' \in W$ and any formula $\mathsf{K}_a^c\varphi \in hd(w)$, if $\delta_a(w, w') \leq c$, then $\varphi \in hd(w')$.*

Proof. Assumptions $\mathsf{K}_a^c\varphi \in hd(w)$ and $\delta_a(w, w') \leq c$, by Lemma 15, imply that $\mathsf{K}_a^{c-\delta_a(w,w')}\varphi \in hd(w')$. Thus, $hd(w') \vdash \varphi$ by the Truth axiom. Therefore, $\varphi \in hd(w')$ due to the maximality of the set $hd(w')$. ⊠

The next technical observation is used in the proof of Lemma 19.

Lemma 18. *If $\neg\mathsf{H}_a^c\varphi \in hd(w)$ where $w \in W$, then set $\{\neg\varphi\}$ is consistent.*

Proof. Assume the opposite. Thus, $\vdash \varphi$. Hence, $\vdash \mathsf{H}_a^c\varphi$ by H-Necessitation inference rule, which contradicts the assumption of the lemma $\neg\mathsf{H}_a^c\varphi \in hd(w)$ due to the consistency of the set $hd(w)$. Therefore, set $\{\neg\varphi\}$ is consistent. ⊠

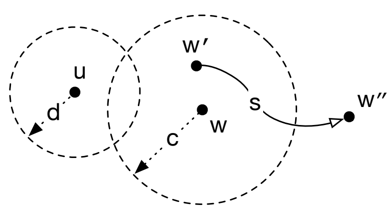

Figure 7: Illustration of Lemma 19.

The following lemma is fundamental to the proof of the completeness. It constructs states w' and w'' that are referred to by item 5 of Definition 6. Note that in the proof of this lemma we choose w' to be state w in all cases except Case III, where the twin node construction is used to specify w'.

Lemma 19. *For any state $w \in W$, any formula $\neg\mathsf{H}_a^c\varphi \in hd(w)$, and any action $(\sigma, u, d) \in D$, there is an action profile $s \in D^{\mathcal{A}}$ and states $w', w'' \in W$ such that $\delta_a(w, w') \leq c$, $s(a) = (\sigma, u, d)$, $(w', s, w'') \in M$, and $\neg\varphi \in hd(w'')$, see Figure 7.*

Proof. Let action profile s be defined as follows:

$$s(x) = \begin{cases} (\sigma, u, d), & \text{if } x = a, \\ (\top, u, 0), & \text{otherwise.} \end{cases} \tag{4}$$

Case I: $\mathsf{H}_a^d\sigma \notin hd(u)$. By Lemma 18, the set $\{\neg\varphi\}$ is consistent. By Lemma 2, there is a maximal consistent extension X of this set. Let w' be state w and w'' be the single-element sequence X. Then, $\delta_a(w, w') = \delta_a(w, w) = 0 \leq c$ by the Identity of Indiscernibles property in Definition 2 and the assumption $c \geq 0$ in Definition 1. Also, $s(a) = (\sigma, u, d)$ by equation (4) and $\neg\varphi \in X = hd(w'')$ by the choice of sequence w'' and set X.

To show that $(w', s, w'') \in M$, consider any agent $x \in \mathcal{A}$, any formula ψ, any state v, and any real number $r \geq 0$. Suppose that $\mathsf{H}_x^r\psi \in hd(v)$, $\delta_x(w', v) \leq r$, and $s(x) = (\psi, v, r)$. By Definition 10, it suffices to show $\psi \in hd(w'')$.

First, we prove that $x \neq a$. Indeed, suppose that $x = a$. Thus, $(\psi, v, r) = s(x) = s(a) = (\sigma, u, d)$ by equation (4). Then, $\psi = \sigma$, $v = u$, and $r = d$. Hence, assumption $\mathsf{H}_x^r\psi \in hd(v)$ implies that $\mathsf{H}_a^d\sigma \in hd(u)$, which contradicts the assumption of the case. Therefore, $x \neq a$.

Hence, $(\psi, v, r) = s(x) = (\top, u, 0)$ by equation (4). Thus, $\psi = \top$. Therefore, $\psi \in hd(w'')$ by the maximality of the set $hd(w'')$.

Case II: $\delta_a(u, w) > d - c$ and $c = 0$. Hence, $\delta_a(u, w) > d$. By Lemma 18, the set $\{\neg\varphi\}$ is consistent. By Lemma 2, this set can be extended to a maximal consistent set X. Choose w' to be state w and w'' to be the single-element sequence X. Note that $\delta_a(w, w') = 0 \leq c$ by the Identity of Indiscernibles property of Definition 2 and the assumption $c = 0$ of the case. Also, $\neg\varphi \in hd(w'')$ by the choice of set X and sequence w''.

To show $(w', s, w'') \in M$, consider any agent $x \in \mathcal{A}$, any formula ψ, any state v, and any real number $r \geq 0$. Suppose that $\mathsf{H}_x^r\psi \in hd(v)$, $\delta_x(w', v) \leq r$, and $s(x) = (\psi, v, r)$. By Definition 10, it suffices to show $\psi \in hd(w'')$. Indeed, if $x = a$, then $v = u$, $\psi = \sigma$ and $r = d$ due to equation (4). Thus, $\delta_a(w, u) = \delta_x(w', v) \leq r = d$, which, by the Symmetry property of

Definition 2, contradicts to $\delta_a(u, w) > d$. Hence, $x \neq a$. Thus, $\psi = \top$ by equation (4). Therefore, $\psi = \top \in hd(w'')$ by the maximality of $hd(w'')$.

Case III: $\delta_a(u, w) > d - c$ and $c > 0$. By Lemma 13, $w_0 = w :: (a, c, 0) :: hd(w)$ and $w_1 = w :: (a, c, 1) :: hd(w)$ are both epistemic states. Hence, by Lemma 14 and the assumption of the case, either $\delta_a(u, w_0) > d$ or $\delta_a(u, w_1) > d$. Without loss of generality, let

$$\delta_a(u, w_0) > d. \tag{5}$$

By Lemma 18, the set $\{\neg\varphi\}$ is consistent. By Lemma 2, this set can be extended to a maximal consistent set X. Choose w' to be state w_0 and w'' to be the single-element sequence X. Note that $\delta_a(w, w_0) = c$ by Definition 8 and the choice of w_0. Also, $\neg\varphi \in hd(w'')$ by the choice of set X and sequence w''.

To show $(w', s, w'') \in M$, consider any agent $x \in \mathcal{A}$, any formula ψ, any state v, and any real number $r \geq 0$. Suppose that $\mathsf{H}^r_x\psi \in hd(v)$, $\delta_x(w', v) \leq r$, and $s(x) = (\psi, v, r)$. By Definition 10, it suffices to show $\psi \in hd(w'')$. Indeed, if $x = a$, then $v = u$, $\psi = \sigma$ and $r = d$ due to equation (4). Thus, $\delta_a(w_0, u) = \delta_x(w', v) \leq r = d$, which, by the Symmetry property of Definition 2, contradicts formula (5). Hence, $x \neq a$. Thus, $\psi = \top$ by equation (4). Therefore, $\psi = \top \in hd(w'')$ by the maximality of $hd(w'')$.

Case IV: $\mathsf{H}^d_a\sigma \in hd(u)$ and $\delta_a(u, w) \leq d - c$. Consider set $\{\neg\varphi, \sigma\}$. First we show that this set is consistent. Suppose the opposite. Thus, $\sigma \vdash \varphi$. Hence, $\vdash \sigma \to \varphi$ by Lemma 1. Then, $\vdash \mathsf{H}^d_a\sigma \to \mathsf{H}^d_a\varphi$, by the Monotonicity inference rule. Thus, $hd(u) \vdash \mathsf{H}^d_a\varphi$ by the Modus Ponens rule and the assumption $\mathsf{H}^d_a\sigma \in hd(u)$ of the case. By the assumption $\delta_a(u, w) \leq d - c$ of the case and because $\delta_a(u, w) \geq 0$ by Definition 2, we have $d \geq c$. It then follows from the Strategic Positive Introspection axiom, by the Modus Ponens inference rule, that $hd(u) \vdash \mathsf{K}^{d-c}_a\mathsf{H}^c_a\varphi$. Thus, $\mathsf{K}^{d-c}_a\mathsf{H}^c_a\varphi \in hd(u)$ by the maximality of the set $hd(u)$. In other words, $\mathsf{K}^{d-c-\delta_a(u,w)+\delta_a(u,w)}_a\mathsf{H}^c_a\varphi \in hd(u)$. Note that $d - c - \delta_a(u, w) \geq 0$ by the assumption $\delta_a(u, w) \leq d - c$ of the case. Thus, $\mathsf{K}^{d-c-\delta_a(u,w)}_a\mathsf{H}^c_a\varphi \in hd(w)$ by Lemma 15. Hence, $hd(w) \vdash \mathsf{H}^c_a\varphi$ by the Truth axiom and the Modus Ponens inference rule. Then, $\neg\mathsf{H}^c_a\varphi \notin hd(w)$ due to the consistency of the set $hd(w)$, which contradicts the assumption of the lemma. Therefore, the set $\{\neg\varphi, \sigma\}$ is consistent.

By Lemma 2, there is a maximal consistent extension X of the set $\{\neg\varphi, \sigma\}$. Let w' be state w and w'' be the single-element sequence X. Then,

$\delta_a(w, w') = \delta_a(w, w) = 0 \leq c$ by the Identity of Indiscernibles property in Definition 2 and because $c \geq 0$ by Definition 1. Also, $s(a) = (\sigma, u, d)$ by equation (4) and $\neg\varphi \in hd(w'') = X$ by the choice of set X and sequence w''.

To show that $(w', s, w'') \in M$, consider any agent $x \in \mathcal{A}$, any formula ψ, any state v, and any real number $r \geq 0$. Suppose that $\mathsf{H}^r_x\psi \in hd(v)$, $\delta_x(w', v) \leq r$, and $s(x) = (\psi, v, r)$. By Definition 10, it suffices to show that $\psi \in hd(w'')$. By equation (4), formula ψ is either σ or $\top$. Note that $\sigma \in X = hd(w'')$ by the choice of set X and $\top \in hd(w'')$ due to the maximality of the set $hd(w'')$. ⊠

Lemma 20. *For all epistemic states $w, w', w'' \in W$, any formula $\mathsf{H}^c_a\varphi \in hd(w)$, and any action profile $s \in D^{\mathcal{A}}$, if $\delta_a(w, w') \leq c$, $(w', s, w'') \in M$, and $s(a) = (\varphi, w, c)$, then $\varphi \in hd(w'')$.*

Proof. Assumption $\delta_a(w, w') \leq c$ implies $\delta_a(w', w) \leq c$ by the Symmetry property in Definition 2. Thus, by Definition 10, assumptions $s(a) = (\varphi, w, c)$, $\mathsf{H}^c_a\varphi \in hd(w)$, and $(w', s, w'') \in M$ imply $\varphi \in hd(w'')$. ⊠

The next lemma combines the results above to connect the membership in the set $hd(w)$ with the satisfiability at state w of the canonical epistemic transition system.

Lemma 21. *$\varphi \in hd(w)$ iff $w \Vdash \varphi$ for each formula $\varphi \in \Phi$ and each state $w \in W$ of the canonical transition system.*

Proof. We prove this statement by induction on the structural complexity of formula φ. If formula φ is a propositional variable, then the required follows from Definition 11 and Definition 6. The cases when formula φ is a negation or an implication follow from Definition 6 and the maximality and the consistency of the set $hd(w)$ in the standard way.

Suppose that formula φ has the form $\mathsf{K}^c_a\psi$.

$(\Rightarrow)$: Consider any epistemic state $w' \in W$ such that $\delta_a(w, w') \leq c$. By Definition 6, it suffices to show that $w' \Vdash \psi$. Indeed, $\psi \in hd(w')$ by Lemma 17. Therefore, $w' \Vdash \psi$ by the induction hypothesis.

$(\Leftarrow)$: Suppose that $\mathsf{K}^c_a\psi \notin hd(w)$. Thus, $\neg\mathsf{K}^c_a\psi \in hd(w)$ due to the maximality of the set $hd(w)$. Hence, by Lemma 16, there is an epistemic state $w' \in W$ such that $\delta_a(w, w') \leq c$ and $\neg\psi \in hd(w')$. Thus, $\psi \notin hd(w')$ due to the consistency of the set $hd(w')$. Hence, $w' \nVdash \psi$ by the induction hypothesis. Therefore, $w \nVdash \mathsf{K}_a^c\psi$ by Definition 6.

Assume now that formula φ has the form $\mathsf{H}_a^c\psi$.

$(\Rightarrow)$: Consider arbitrary epistemic states $w', w'' \in W$ and an action profile $s \in D^{\mathcal{A}}$ such that $\delta_a(w, w') \leq c$, $(w', s, w'') \in M$, and $s(a) = (\psi, w, c)$. By Definition 6, it suffices to show that $w'' \Vdash \psi$. Indeed, $\psi \in hd(w'')$ by Lemma 20. Therefore, $w'' \Vdash \psi$ by the induction hypothesis.

$(\Leftarrow)$: Suppose that $w \Vdash \mathsf{H}_a^c\psi$. Thus, by Definition 6, there is an action $(\sigma, u, d) \in D$ such that for any $w', w'' \in W$ and any action profile $s \in D^{\mathcal{A}}$ if $\delta_a(w, w') \leq c$, $s(a) = (\sigma, u, d)$, and $(w', s, w'') \in M$, then $w'' \Vdash \psi$.

Assume now that $\mathsf{H}_a^c\psi \notin hd(w)$. Thus, $\neg\mathsf{H}_a^c\psi \in hd(w)$ due to the maximality of the set $hd(w)$. Hence, by Lemma 19, there is an action profile $s \in D^{\mathcal{A}}$ and epistemic states $w_0', w_0'' \in W$ such that $\delta_a(w, w_0') \leq c$, $s(a) = (\sigma, u, d)$, $(w_0', s, w_0'') \in M$, and $\neg\psi \in hd(w_0'')$. Thus, $\psi \notin hd(w_0'')$ due to the consistency of the set $hd(w_0'')$. Then, $w_0'' \nVdash \psi$ by the induction hypothesis. Let $w' = w_0'$ and $w'' = w_0''$. Then, $w_0'' \Vdash \psi$ by the choice of action $(\sigma, u, d) \in D$, which yields a contradiction. ⊠

Now we are ready to state and prove the strong completeness theorem for our logical system with respect to the class of arbitrary (not necessarily with finite metrics) epistemic transition systems.

Theorem 1. *If $X \nvdash \varphi$, then there is an epistemic state w of an epistemic transition system such that $w \Vdash \chi$ for each formula $\chi \in X$ and $w \nVdash \varphi$.*

Proof. Suppose that $X \nvdash \varphi$. Hence, the set $X \cup \{\neg\varphi\}$ is consistent. By Lemma 2, there is a maximal consistent extension X_0 of the set $X \cup \{\neg\varphi\}$. Let w_0 be a single-element sequence consisting of set X_0. By Definition 7, sequence w_0 is an epistemic state of the canonical epistemic transition system. Note that $hd(w_0) = X_0$. Then, $w_0 \Vdash \neg\varphi$ and $w_0 \Vdash \chi$ for each formula $\chi \in X$ by Lemma 21 and the choice of set X_0. Therefore, $w_0 \nVdash \varphi$ by Definition 6. ⊠

7. Weak Completeness for Finite Metrics

The notion of a finite metric space is much more commonly used and usually is referred to as just "metric space" [50]. In this section, we first show how to convert an epistemic transition system to a system with finite metrics.

To achieve this task, we utilize a technique from the metric space theory called "truncation" [60]. Then, we state and prove the weak completeness theorem for epistemic transition systems with finite metrics.

Definition 12. *For any metric space (W, δ), any elements $u, v \in W$, and any positive ("threshold") real number t, let* ***truncated distance*** $\delta\lfloor_t$ *be*

$$\delta\lfloor_t(u,v) = \begin{cases} \delta(u,v), & \textit{if } \delta(u,v) \leq t, \\ t, & \textit{otherwise.} \end{cases}$$

Lemma 22. *$(X, \delta\lfloor_t)$ is a finite metric space for each metric space (X, δ) and each positive real threshold value t.*

Proof. The Identity of Indiscernibles and Symmetry properties for the truncated metric follow from Definition 12 and Definition 2. To prove the Triangle Inequality, consider any $u, v, w \in W$. We show that

$$\delta\lfloor_t(u,v) \leq \delta\lfloor_t(u,w) + \delta\lfloor_t(w,v).$$

Case I: $\max\{\delta(u,w), \delta(w,v)\} \geq t$. Hence, by Definition 12,

$$\max\{\delta\lfloor_t(u,w), \delta\lfloor_t(w,v)\} = t.$$

Thus, by Definition 12,

$$\delta\lfloor_t(u,v) \leq t = \max\{\delta\lfloor_t(u,w), \delta\lfloor_t(w,v)\} \leq \delta\lfloor_t(u,w) + \delta\lfloor_t(w,v).$$

Case II: $\max\{\delta(u,w), \delta(w,v)\} < t$. Then, by Definition 12,

$$\delta\lfloor_t(u,w) + \delta\lfloor_t(w,v) = \delta(u,w) + \delta(w,v).$$

Hence, by Definition 12 and the Triangle Inequality property for metric δ,

$$\delta\lfloor_t(u,v) \leq \delta(u,v) \leq \delta(u,w) + \delta(w,v) = \delta\lfloor_t(u,w) + \delta\lfloor_t(w,v).$$

$\boxtimes$

Let $rank(\varphi)$ be the largest modality superscript that appears inside formula φ. For example, $rank(\mathsf{K}_a^2\mathsf{H}_b^3 p) = 3$.

Lemma 23. *If $\Vdash$ is the satisfiability relation of an epistemic transition system $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$ and $\Vdash'$ is the satisfiability relation of the system $(W, \{\delta_a\lfloor_t\}_{a\in\mathcal{A}}, D, M, \pi)$ where $t > 0$, then $w \Vdash \varphi$ if and only if $w \Vdash' \varphi$, for any formula $\varphi \in \Phi$ such that $rank(\varphi) < t$.*

Proof. We prove the lemma by induction on the structural complexity of formula φ. If formula φ is a propositional variable p, then, by Definition 6, both $w \Vdash p$ and $w \Vdash' p$ are equivalent to statement $w \in \pi(p)$. Thus, $w \Vdash p$ if and only if $w \Vdash' p$. The cases when φ is a negation or an implication follow from the induction hypothesis and Definition 6.

Assume that φ has the form $\mathsf{K}^c_a\psi$. Then assumption $rank(\mathsf{K}^c_a\psi) < t$ implies that $c < t$ and $rank(\psi) < t$.

$(\Rightarrow)$: Suppose $w \nVdash' \mathsf{K}^c_a\psi$. Thus, by Definition 6, there is $u \in W$ such that $\delta_a\lfloor_t(w,u) \le c$ and $u \nVdash' \psi$. Then, $\delta_a(w,u) \le c$ due to $c < t$ and Definition 12. Also, by the induction hypothesis, $u \nVdash \psi$. Therefore, $w \nVdash \mathsf{K}^c_a\psi$ by Definition 6.

$(\Leftarrow)$: Assume that $w \nVdash \mathsf{K}^c_a\psi$. Thus, by Definition 6, there is $u \in W$ such that $\delta_a(w,u) \le c$ and $u \nVdash \psi$. Then, $\delta_a\lfloor_t(w,u) \le \delta_a(w,u) \le c$ by Definition 12. Also, by the induction hypothesis, $u \nVdash' \psi$. Therefore, $w \nVdash' \mathsf{K}^c_a\psi$ by Definition 6.

Suppose that φ has the form $\mathsf{H}^c_a\psi$. Then assumption $rank(\mathsf{H}^c_a\psi) < t$ implies that $c < t$ and $rank(\psi) < t$.

$(\Rightarrow)$: Suppose that $w \nVdash' \mathsf{H}^c_a\psi$. Consider an arbitrary action $\alpha \in D$. Then, by Definition 6, there are states $u, v \in W$ and an action profile $s \in D^{\mathcal{A}}$ such that $\delta_a\lfloor_t(w,u) \le c$, $s(a) = \alpha$, $(u,s,v) \in M$, and $v \nVdash' \psi$. Note that $\delta_a(w,u) \le c$ due to $c < t$ and Definition 12. Also, by the induction hypothesis, $v \nVdash \psi$. Hence, for any action $\alpha \in D$, there are states $u, v \in W$ and an action profile $s \in D^{\mathcal{A}}$ such that $\delta_a(w,u) \le c$, $s(a) = \alpha$, $(u,s,v) \in M$, and $v \nVdash \psi$. Therefore, $w \nVdash \mathsf{H}^c_a\psi$ by Definition 6.

$(\Leftarrow)$: Suppose that $w \nVdash \mathsf{H}^c_a\psi$. Consider an arbitrary action $\alpha \in D$. Then, by Definition 6, there are states $u, v \in W$ and an action profile $s \in D^{\mathcal{A}}$ such that $\delta_a(w,u) \le c$, $s(a) = \alpha$, $(u,s,v) \in M$ and $v \nVdash \psi$. Note that $\delta_a\lfloor_t(w,u) \le \delta_a(w,u) \le c$ by Definition 12. Also, by the induction hypothesis, $v \nVdash' \psi$. Hence, for any action $\alpha \in D$, there are states $u, v \in W$ and an action profile $s \in D^{\mathcal{A}}$ such that $\delta_a\lfloor_t(w,u) \le c$, $s(a) = \alpha$, $(u,s,v) \in M$, and $v \nVdash' \psi$. Therefore, $w \nVdash' \mathsf{H}^c_a\psi$ by Definition 6. ⊠

Next, we state and prove the weak completeness theorem for the epistemic transition systems with finite metrics.

Theorem 2. *If $w \Vdash \varphi$ for every epistemic state of every epistemic transition system with finite metrics, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. By Theorem 1, there is an epistemic state w of an epistemic transition system $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$ such that $w \nVdash \varphi$. Choose any real number $t > rank(\varphi)$. Let $\Vdash'$ be the satisfiability relation for the epistemic transition system with finite metrics $(W, \{\delta_a \downharpoonleft_t\}_{a\in\mathcal{A}}, D, M, \pi)$. Then, $w \nVdash' \varphi$ by Lemma 23. $\boxtimes$

8. Incompleteness for Finite Metrics

Theorem 1 shows the *strong* completeness of our logical system with respect to the class of arbitrary epistemic transition systems. Theorem 2 establishes only the *weak* completeness with respect to the class of epistemic transition systems with finite metrics. As Theorem 3 below shows, not only the strong completeness for the systems with finite metrics does not hold for our logical system, but there is no strongly sound logical system for which it does.

Definition 13. *A logical system $\mathcal{L}$ is **strongly sound** with respect to epistemic transition systems when for any set of formulae X, any formula φ such that $X \vdash_{\mathcal{L}} \varphi$, and any epistemic state w of any epistemic transition system, if $w \Vdash \chi$ for each $\chi \in X$, then $w \Vdash \varphi$.*

Theorem 3. *For any strongly sound logical system $\mathcal{L}$ with respect to epistemic transition systems, there is a set of formulae $X \subseteq \Phi$ and a single formula $\varphi \in \Phi$ such that*

1. *for each epistemic state w of each epistemic transition system with finite metrics, if $w \Vdash \chi$ for each formula $\chi \in X$, then $w \Vdash \varphi$,*
2. *$X \nvdash_{\mathcal{L}} \varphi$.*

Proof. Recall that our language has at least one propositional variable. Let p be such a variable and c be an arbitrary positive real number. Consider an infinite set of formulae $X = \{\mathsf{K}^1_a p, \mathsf{K}^2_a p, \mathsf{K}^3_a p, \dots\}$ and formula $\varphi = \mathsf{H}^c_a p$. We will show that statements 1 and 2 hold for the chosen set X and formula φ.

To prove statement 1, consider an epistemic state $w \in W$ of an epistemic transition system $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$ with finite metrics. We show that if $w \Vdash \mathsf{K}^n_a p$ for each integer $n \geq 1$, then $w \Vdash \mathsf{H}^c_a p$. Suppose that $w \nVdash \mathsf{H}^c_a p$. By Definition 4, set D contains at least one action α. Thus, by Definition 6, assumption $w \nVdash \mathsf{H}^c_a p$ implies that there are $w', w'' \in W$ and $s \in D^{\mathcal{A}}$ such that $\delta_a(w, w') \leq c$, $s(a) = \alpha$, $(w', s, w'') \in M$, and $w'' \nVdash p$. Because metric δ_a is assumed to be finite, there must exist an integer m such that $\delta_a(w, w'') \leq m$. By Definition 6, statements $\delta_a(w, w'') \leq m$ and $w'' \nVdash p$ imply that $w \nVdash \mathsf{K}^m_a p$, which contradicts our assumption that $w \Vdash \mathsf{K}^n_a p$ for each $n \geq 1$.

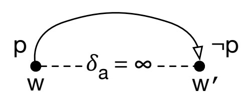

Figure 8: Towards the Proof of Incompleteness.

To show statement 2, recall that our logical system is strongly sound with respect to the epistemic transition systems. Thus, by Definition 13, to prove that $X \nvdash \varphi$, it suffices to find an epistemic state $w \in W$ of an epistemic transition system $(W, \{\delta_a\}_{a\in\mathcal{A}}, D, M, \pi)$ *with possibly infinite metrics* such that $w \Vdash \mathsf{K}^n_a p$ for each integer $n \geq 1$ and $w \nVdash \mathsf{H}^c_a p$. An example of such a transition system is depicted in Figure 8. It consists of states w and w', where $\delta_a(w, w') = \infty$. Propositional variable p holds in state w, but not in state w'. Then, $w \Vdash \mathsf{K}^n_a p$ for each integer $n \geq 1$ by Definition 6. At the same time, let the mechanism of the system be such that from state w the system transitions to state w' under any action profile of the agents. Therefore, $w \nVdash \mathsf{H}^c_a p$ by Definition 6. $\boxtimes$

Note that the proof of Theorem 3 uses the fact that our language Φ contains *two types of modalities*: K and H. However, the strong completeness does not hold even if we restrict the formulae in set $X \cup \{\varphi\}$ to only those that contain just the modality K with at least *two agents*. Indeed, one can consider an infinite set of formulae $X = \{\mathsf{K}^1_a p, \mathsf{K}^2_a p, \mathsf{K}^3_a p, \dots\}$ and a formula $\varphi = \mathsf{K}^c_b p$, where a and b are two distinct agents. The proof of Theorem 3 for this choice of X and φ is similar to the one given above. We believe that if the language is further restricted to just modality K_a for a single fixed agent a, then the strong completeness with respect to the finite metrics semantics would hold.

9. Conclusion

The contributions of this article are as follows. First, we introduced the notion of a know-how strategy under uncertainty as a strategy that can be used not only at a given state, but at any state within a given distance from the given state. Second, we proposed a sound logical system that describes the interplay between the know-how under uncertainty and the knowledge modalities. We proved the strong completeness of this system with respect to arbitrary transition systems and the weak completeness with respect to transitions systems with finite metrics. We also showed that the strong completeness with respect to the systems with finite metrics does not hold.

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