A Modal Logic for Reasoning about Economic Policies

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Abstract

The article introduces a modal logic for reasoning about combined effect of economic policies imposed on a group of rational agents. Modalities in this language are labeled by policies applied to the players in a strategic game. The resulting logical system allows to reason about properties that are true in all Nash equilibria of the game modified by a specific policy. The main technical result is the completeness theorem for the proposed logical system.

1 Introduction

Motivation. Dynamic logic [7] provides a unified framework for reasoning about outcomes of actions applied to different systems. In this article we propose a framework for reasoning about outcomes of different economic policies imposed on rational agents. At the core of our framework is the modal operator $\Box_p\varphi$ which is intended to capture the statement "if economic policy p is imposed on the rational agents, then statement φ will be true in all Nash equilibria of the resulting system"[1].

As an example, consider a situation when government evaluates two alternative tools to stimulate economy: a policy that lowers the tax rate by 1% and a policy that lowers the federal discount rate by 1%. Of course, the government can consider policies that use a combination of these two policies. For example, by policy $(2, 3)$ we mean lowering the tax rate by 2% and lowering the discount rate by 3%. In this article we introduce a modal language to describe the effect

[1] Note that if policy p results in a game that has no Nash equilibria, then statement $\Box_p\varphi$ is vacuously true.

produced by different policies. For example, the statement

$$
\begin{aligned}
&\text{“unemployment rate is below 10\%”} \rightarrow \\
&\quad \Box_{(2,3)} \text{“unemployment rate is below 5\%”}
\end{aligned} \tag{1}
$$

says that if the current unemployment rate is below 10%, then after the tax rate is lowered by 2% and the discount rate is lowered by 3%, the unemployment rate is *guaranteed* to be under 5%. The statement

$$
\begin{aligned}
&\text{“unemployment rate is below 10\%”} \rightarrow \\
&\quad (\neg\Box_{(2,0)}\neg \text{“unemployment rate is below 5\%”} \\
&\qquad \wedge \neg\Box_{(2,0)} \text{“unemployment rate is below 5\%”})
\end{aligned} \tag{2}
$$

says that if the current unemployment rate is below 10%, then reducing the interest rate alone by 2% *might* result in an unemployment rate under 5%, but one can not guarantee this. The next statement,

$$
\begin{aligned}
&\Box_{(2,0)} (\text{“unemployment rate is below 7\%”} \rightarrow \\
&\quad \Box_{(0,3)} \text{“unemployment rate is below 5\%”})
\end{aligned} \tag{3}
$$

says that if 2% tax reduction alone can lower the unemployment rate to under 7%, then the decrease in the discount rate by additional 3% would further reduce the unemployment rate to under 5%.

Game Formalism. The formal semantics of our logical system is based on Nash equilibria of multi-player strategic games. Our logical system studies the validity of statements in Nash equilibria of a game. We interpret different policies as adjustments to the utility functions of the game. As an example, consider the utility functions u_a and u_b of the Prisoner's Dilemma game depicted in Table 1. Each of the strategies a_1 and b_1 is commonly referred to as "cooperation"

$\{u_a, u_b\}$	b_1	b_2
a_1	-2,-2	-6,0
a_2	0,-6	-4,-4

Table 1: Utility functions for the Prisoner's Dilemma game.

and each of the strategies a_2 and b_2 as "defection". The state of a game could be specified by the current set of utility functions and the current Nash equilibrium of the game. The strategy profile (a_2, b_2) is the only Nash equilibrium of the Prisoner's Dilemma game. In this equilibrium both players have equal penalties imposed on them. We formally write this as

$$
(\{u_a, u_b\}, \langle a_2, b_2 \rangle) \vDash \text{“the players have equal penalties”} \tag{4}
$$

Consider a policy (the adjustments to the utility functions) δ specified in Table 2. If these adjustments are applied to the utility functions of the Prisoner's

δ	b_1	b_2
a_1	1,-1	-1,1
a_2	-1,1	1,-1

Table 2: Policy (the set of adjustments) δ.

$\{u'_a, u'_b\}$	b_1	b_2
a_1	-1,-3	-7,1
a_2	-1,-5	-3,-5

Table 3: Utility functions adjusted by policy δ.

Dilemma game, then the modified game, depicted in Table 3, has two Nash equilibria: (a_2, b_1) and (a_2, b_2). In both of these equilibria the penalty imposed on the first player is less than the penalty imposed on the second player. We formally write this as

$$(\{u_a, u_b\}, \langle a_2, b_2\rangle) \vDash \Box_\delta(\text{“the first player has a smaller penalty”}) \tag{5}$$

Note that if policy δ is applied to the Prisoner's Dilemma game twice, then the

$\{u''_a, u''_b\}$	b_1	b_2
a_1	0,-4	-8,2
a_2	-2,-4	-2,-6

Table 4: Utility functions adjusted by policy 2δ.

resulting game, depicted in Table 4, has no Nash equilibria and, thus, statement

$$(\{u_a, u_b\}, \langle a_2, b_2\rangle) \vDash \Box_{2\delta}(\text{“the first player has a smaller penalty”}) \tag{6}$$

is vacuously true.

Adjustment δ can be applied to the set of the utility functions not only any positive integer number of times, but we can also consider adjustments of the form $p\delta$, where p is an arbitrary, possibly negative, real number. The set of all such adjustments is closed with respect to compositions in the sense that the adjustments $p\delta$ and $q\delta$, applied together, are equivalent to adjustment $(p+q)\delta$. If we only consider policies of the form $p\delta$, then each such policy could be identified with the real number p and the composition operation on policies would correspond to the addition of real numbers. In such situations we write $\Box_p\varphi$ instead of $\Box_{p\delta}\varphi$ and say that the number p is the name of the policy $p\delta$.

As another example, consider two possible adjustments δ_1 and δ_2 to the set of the utility functions. Any linear combination $p_1\delta_1 + p_2\delta_2$ of these adjustments could also be viewed as a policy. Each such policy $p_1\delta_1 + p_2\delta_2$ could be identified with (or named by) the pair $p = (p_1, p_2) \in \mathbb{R}^2$. The composition of two such policies corresponds to the pair-wise (vector) addition in $\mathbb{R}^2$.

Adjustments to the set of utility functions do not have to be as simple as addition or subtraction of certain values. Consider, for example, seven players named 0, 1, 2, $\dots$, 6 sitting around a dining table. Each player i has her own utility function u_i. Possible adjustments to the set of utility functions could be shifting the functions around the table by k positions:

$$u_i^k(\langle s_0, \dots, s_6 \rangle) = u_{i'}(\langle s_0, \dots, s_6 \rangle), \tag{7}$$

where $i' \equiv k + i \pmod 7$. There are exactly seven such adjustments that can be identified with (or named by) elements of $\mathbb{Z}_7$. The composition of these adjustments corresponds to the addition operation in $\mathbb{Z}_7$.

Policy Groups. Above we have looked at examples of sets of policies that can be naturally identified with (or named by) elements of $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{Z}_7$. In all cases, the composition of policies would correspond to the addition operation on the corresponding name space. We distinguish name spaces of policies and the actual policies. To keep the presentation as general as possible, we assume that the name space of policies is a triple consisting of an arbitrary set P, a fixed element $e \in P$, and an abstract operation $*$ that satisfies the following properties:

1. $p * (q * r) = (p * q) * r$, for all $p, q, r \in P$,
2. $p * e = e * p = p$, for all $p \in P$,
3. for every $p \in P$ there is $p^{-1} \in P$ such that $p * p^{-1} = p^{-1} * p = e$.

In abstract algebra, triple $\langle P, e, * \rangle$ is called a group. We call it a *policy group*. Triples $\langle \mathbb{R}, 0, + \rangle$, $\langle \mathbb{R}^2, (0,0), + \rangle$, and $\langle \mathbb{Z}_7, [0], + \rangle$ are examples of different policy groups.

Policy groups are name spaces that we use to identify different policies. They are not the policies themselves. When specifying the semantics of our logical system, we assign a specific policy (an adjustment to the set of utility functions) to each element of the appropriate policy group. Once this assignment is chosen, statement $\Box_p \varphi$ is interpreted as "after the policy (adjustment) named by p is applied to the utility functions of players in the current game, the statement φ is true in all Nash equilibria of the resulting game". If policy p results in a game that has no Nash equilibria, then statement $\Box_p \varphi$ is vacuously true. Further details of this semantics are given in Definition 3.

For the sake of simplicity, in what follows, policy names are referred to as simply "policies" when there is no confusion.

Modal Properties. Statements (1), (2), and (3) might be true or false depending on the current economic situation. Claims (4), (5), and (6) could be true or false depending on the choice of the utility functions u_a and u_b as well as on the choice of the policy assigned to the policy name p.

In this article we study universal properties of policies that are true in all Nash equilibria of all strategic games based on a given policy group $\langle P, e, *\rangle$. A trivial example of such a universal property is

$$\Box_e\varphi \to \varphi. \tag{A1}$$

This property states that if statement φ is true in all Nash equilibria of the non-adjusted game, then it is true in the current equilibrium as well, because, as we will see later, policy e corresponds to no adjustments to the set of utility functions.

Another such property is

$$\Box_{p*q}\varphi \to \Box_p\Box_q\varphi. \tag{A2}$$

The assumption of this statement is that property φ is true in each Nash equilibria after policies p and q are applied. The conclusion states that if only policy p is applied, then statement φ will become true in each Nash equilibrium after additional policy q is introduced. Note that the converse of the above property is not true. Indeed, if policy p results in a game that has no Nash equilibria at all, then statement $\Box_p\Box_q\varphi$ is vacuously true. However, the combined policy $p*q$ might have a Nash equilibrium in which statement φ is false. The converse does become true if we also assume the existence of at least one Nash equilibrium under policy p:

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_q\varphi \to \Box_{p*q}\varphi). \tag{8}$$

A less trivial example of a universal policy property is

$$\neg\Box_{p*q}\varphi \to \Box_p\neg\Box_q\varphi. \tag{A3}$$

It states that if the combined policy $p*q$ *might* result in statement φ being false, then after policy p is applied, an additional application of policy q might result in statement φ being false too.

Under our formal semantics that we introduce in the next section, policy named by p^{-1} will be the adjustment to the set of utility functions that retracts changes made by policy p. Thus, one might expect the following statement to be universally true $\Box_p\Box_{p^{-1}}\varphi \to \varphi$. However, this statement is not necessarily true because if policy p results in a game that has no Nash equilibria at all, then statement $\Box_p\Box_{p^{-1}}\varphi$ is vacuously true while statement φ may be false. Just like it has been done in the case of statement (8), this issue could be addressed by adding assumption $\neg\Box_p\bot$:

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_{p^{-1}}\varphi \to \varphi). \tag{9}$$

When modified this way, the statement indeed becomes universally true.

In general, the composition of adjustments to the set of utility functions is not commutative. To illustrate this, consider a game between Alice and six other players sitting around a dining table. Suppose that all players have utility functions that are equal to zero on all strategy profiles. If policy p increases Alice's payoff by £5 and policy q shifts utility functions around the table by one position as defined in (7), then with composition $q*p$ Alice gets £5 and with composition $p*q$ she does not. In our language, we can consider properties of commutative policies by simply assuming that the policy group is commutative. Such groups are called Abelian in abstract algebra. We call them *Abelian policy groups*. The following property is true for all Abelian policy groups:

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_q\varphi \to \Box_q\Box_p\varphi). \tag{10}$$

In this article we give a complete axiomatization of all universal modal properties of policies. Perhaps surprisingly, our logical system consists of only four modal axioms. It includes axioms (A1), (A2), (A3), the distributivity axiom $\Box_p(\varphi \to \psi) \to (\Box_p\varphi \to \Box_p\psi)$, called axiom (A4), and propositional tautologies, as well as Modus Ponens and Necessitation inference rules. In Section 4, we prove that properties (8) and (9) are derivable in our system for any policy group and property (10) is derivable for Abelian policy groups.

Related Works. The axioms of our logical system are axioms of the modal logic S5 [5] with modalities labeled by policies. This is not a coincidence because Nash equilibria in our setting could be viewed as Kripke worlds, thus connecting our work to papers on the logic of public announcements [2] and the dynamic epistemic logic [4]. A variety of approaches to combine modal logic with game theory has been discussed in the literature before [3, 14, 10, 9]. See van der Hoek and Pauly's chapter [15] in the Handbook of Modal Logic for more works in this area. None of these works considers policies and operations on them.

Wooldridge et al. [16] investigated influence of taxation on the preferences of players in Boolean games. Although their setting is similar to ours, the focus of their paper is on designing taxation schemes and on the complexity of algorithms for finding such designs. They do not introduce any modal logical systems for reasoning about taxation.

In their work on reasoning about social choice functions, Troquard et al. [13] considered a social choice function as a special form of strategic games. They proposed a complete logical system for reasoning about such games. The language of their system has modality $\Diamond_C\varphi$ that stands for "coalition C can force φ if players outside coalition C hold on to their current strategy" and modality $\blacklozenge_i\varphi$ that stands for "i locally (at the current reported profile) considers a reported profile where φ is true at least as preferable". Our work fundamentally differs from theirs as we consider modalities labeled by policies (adjustments to the set of utility functions).

Harrenstein et al. [8] proposed a modal logic for reasoning about Nash equilibria and proved its completeness. Their logic has three types of modalities intended to capture the following meanings: (i) a statement holds in all states at least as preferable to a given player as the present one, (ii) if all players choose their prescribed strategies, then the game ends in a situation in which a given statement holds, and (iii) a given statement holds in all states that can be reached if all but the given player are not allowed to deviate from the given set of strategies. Their logical system has an expressive language and a non-trivial set of axioms. However, they do not consider modalities that corresponds to adjustments to utility functions.

Although in this article we focus on economic policies that can be expressed as adjustments to the set of utility functions, one can also consider policies that are constraints on possible behaviours of agents. Ågotnes et al. [1] described a logical system, based on the temporal logic CTL, for reasoning about compliance and different types of robustness in their non-game-theoretic setting. Halpern and Weissman [6] investigated the use of first-order logic for reasoning about non-economic policies, while Pucella and Weissman [12] used dynamic modal logic in a similar setting.

Outline. The article is structured as follows. Section 2 introduces the formal syntax and the game semantics of our logical system. In Section 3, we give axioms of our system. In Section 4, we formally prove some of the statements discussed in the introduction. In Section 5, we prove the soundness of our logical system with respect to the game semantics. Section 6 is dedicated to the proof of the completeness. We first introduce an auxiliary Kripke semantics for our system and prove its completeness with respect to this semantics. Next, we prove the completeness with respect to the original game semantics by showing how a Kripke model can be modified into a set of policies on a strategic game. Section 7 concludes by discussing possible extensions of our logical system for the other types of policies.

2 Syntax and Semantics

In this section we define the language of our formal system and give the precise definition of the policy semantics in terms of strategic games. Throughout the whole article we fix a nonempty set V of propositional variables that represent atomic statements about strategy profiles. An example of such a statement from the Prisoner's Dilemma game in the introduction is "the first player has a smaller penalty". Next we formally define the language of our logical system.

Definition 1 *For any set P, let the language $\Phi(P)$ be defined as follows:*

$$\varphi ::= v \mid \varphi \to \varphi \mid \neg\varphi \mid \Box_p\varphi,$$

where $v \in V$ and $p \in P$.

As usual, we assume that $\bot$ is an abbreviation for $\neg(v \to v)$ for some propositional variable $v \in V$.

We assign a utility function to each policy name. The utility function of player i under policy p is denoted by u_i^p. This assignment provides formal semantics of policies.

A strategic game is usually specified by a set of players, a set of possible strategies for each player, and a utility function for each player. To incorporate policies, our game definition includes utility function u_i^p for each policy name p and each player i. It also includes function ℓ that specifies semantics of propositional variables.

Definition 2 *For any given policy group $\langle P, e, *\rangle$, a tuple*

$$\langle N, \{A_i\}_{i\in N}, \{u_i^p\}_{i\in N}^{p\in P}, \ell\rangle$$

is called a multi-policy game (or just "game") if

1. *N is a finite set (of "players"),*
2. *A_i is a set of strategies or actions available to player i, for each $i \in N$,*
3. *u_i^p is the utility function from set S into real numbers, for each policy $p \in P$ and each $i \in N$, where S is set $\prod_{i\in N} A_i$ whose elements are called strategy profiles,*
4. *ℓ is a function from the set of propositional variables V into subsets of $S = \prod_{i\in N} A_i$.*

As commonly defined in the game theory literature [11, p.14], by a Nash equilibrium of a strategic game $\langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ we mean a pure strategy profile in which no player can unilaterally improve her payoff. In particular, we do not assume that the equilibrium is strict.

For any multi-policy game $G = \langle N, \{A_i\}_{i\in N}, \{u_i^p\}_{i\in N}^{p\in P}, \ell\rangle$ and any $q \in P$, consider the strategic game $\langle N, \{A_i\}_{i\in N}, \{u_i^q\}_{i\in N}\rangle$. By $NE(G, q)$ we mean the set of all (pure) Nash equilibria of this strategic game.

The next definition is the key definition of this article. It provides the formal semantics for our modal operator. Since under any given policy the system might have multiple equilibria, the semantics specifies the satisfiability relation $(q, f) \vDash \varphi$, where $q \in P$ is the *current* policy and $f \in NE(G, q)$ is the Nash equilibrium of the game in which the multiagent system is at the *current* moment.

Definition 3 *For any policy group $\langle P, e, *\rangle$, any formula $\varphi \in \Phi(P)$, any game $G = \langle N, \{A_i\}_{i\in N}, \{u_i^q\}_{i\in N}^{q\in P}, \ell\rangle$, any policy $p \in P$, and any $f \in NE(G, p)$, let the satisfiability relation $(p, f) \vDash_G \varphi$ be defined as follows:*

1. *$(q, f) \vDash_G v$ if $f \in \ell(v)$,*
2. *$(q, f) \vDash_G \neg\varphi$ if $(p, f) \nvDash_G \varphi$,*
3. *$(q, f) \vDash_G \varphi \to \psi$ if $(p, f) \nvDash_G \varphi$ or $(p, f) \vDash_G \psi$,*
4. *$(q, f) \vDash_G \Box_p\varphi$ if $(q * p, h) \vDash_G \varphi$ for each $h \in NE\,(G, q * p)$.*

Note that if set $NE(G, q * p)$ in the item 4 above is empty, then statement $(q * p, h) \vDash_G \varphi$ is vacuously true. We omit the subscript G in the relation $\vDash_G$, when the value of this subscript is clear from the context.

3 Axioms and Rules

For any fixed policy group $\langle P, e, * \rangle$, our logical system consists of the following axioms for each $p, q \in P$ and each $\varphi, \psi \in \Phi(P)$:

(A1) $\Box_e \varphi \to \varphi$,

(A2) $\Box_{p*q}\varphi \to \Box_p \Box_q \varphi$,

(A3) $\neg\Box_{p*q}\varphi \to \Box_p \neg\Box_q \varphi$,

(A4) $\Box_p(\varphi \to \psi) \to (\Box_p \varphi \to \Box_p \psi)$.

We write $\vdash_{\langle P,e,*\rangle} \varphi$ if formula φ is provable from the propositional tautologies and the above axioms using Modus Ponens and Necessitation inference rules:

$$\frac{\varphi, \quad \varphi \to \psi}{\psi} \qquad\qquad \frac{\varphi}{\Box_p \varphi}.$$

We write $X \vdash_{\langle P,e,*\rangle} \varphi$ if formula φ is provable from propositional tautologies, the theorems of our system, and the additional set of axioms X using only the Modus Ponens inference rule. We omit the subscript $\langle P, e, * \rangle$ when its meaning is clear from the context.

4 Examples of Proofs

We prove the soundness of our logical system in the next section. In this section we formally prove in our system properties (8), (9), and (10) from the introduction.

Proposition 1 *$\vdash \neg\Box_p \bot \to (\Box_p \Box_q \varphi \to \Box_{p*q}\varphi)$, for each policy group $\langle P, e, * \rangle$.*

Proof. Note that formula $\neg\Box_q \varphi \to (\Box_q \varphi \to \bot)$ is a propositional tautology. Thus, by the Necessitation inference rule,

$$\vdash \Box_p(\neg\Box_q \varphi \to (\Box_q \varphi \to \bot)).$$

Hence, by axiom (A4) and the Modus Ponens inference rule,

$$\vdash \Box_p \neg\Box_q \varphi \to \Box_p(\Box_q \varphi \to \bot).$$

Thus, using axiom (A4) and the propositional reasoning,

$$\vdash \Box_p \neg\Box_q \varphi \to (\Box_p \Box_q \varphi \to \Box_p \bot).$$

Note that $\neg\Box_{p*q}\varphi \to \Box_p \neg\Box_q \varphi$ is an instance of axiom (A3). Hence, by the propositional reasoning,

$$\vdash \neg\Box_{p*q}\varphi \to (\Box_p \Box_q \varphi \to \Box_p \bot).$$

Then, again by the propositional reasoning, $\vdash \neg\Box_p \bot \to (\Box_p \Box_q \varphi \to \Box_{p*q}\varphi)$. ⊠

Proposition 2 $\vdash \neg\Box_p\bot \to (\Box_p\Box_{p^{-1}}\varphi \to \varphi)$, *for each policy group* $\langle P, e, * \rangle$.

Proof. By Proposition 1,

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_{p^{-1}}\varphi \to \Box_{p*p^{-1}}\varphi).$$

Note that $p * p^{-1} = e$ due to properties of the group operation $*$. Thus,

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_{p^{-1}}\varphi \to \Box_e\varphi).$$

Therefore, by axiom (A1) and the propositional reasoning,

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_{p^{-1}}\varphi \to \varphi).$$

⊠

Proposition 3 $\vdash \neg\Box_p\bot \to (\Box_p\Box_q\varphi \to \Box_q\Box_p\varphi)$, *for each Abelian policy group* $\langle P, e, * \rangle$.

Proof. By Proposition 1,

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_q\varphi \to \Box_{p*q}\varphi).$$

Thus, due to the commutativity of operation $*$ in Abelian groups,

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_q\varphi \to \Box_{q*p}\varphi).$$

Therefore, by axiom (A2) and the propositional reasoning,

$$\vdash \neg\Box_p\bot \to (\Box_p\Box_q\varphi \to \Box_q\Box_p\varphi).$$

⊠

5 Soundness

In this section we prove the soundness of our logical system with respect to the game semantics. The soundness of propositional tautologies and the Modus Ponens rule is straightforward. Below we show the soundness of each of the remaining axioms and of the Necessitation rule as a separate lemma. In what follows, we assume that $\langle P, e, * \rangle$ is a policy group and $G = \langle N, \{A_i\}_{i \in N}, \{u_i^p\}_{i \in N}^{p \in P}, \ell \rangle$ is a multi-policy game. Also, let $q \in P$, $f \in NE(G, q)$, and $\varphi, \psi \in \Phi(P)$.

Lemma 1 *If* $(q, f) \vDash \Box_e\varphi$, *then* $(q, f) \vDash \varphi$.

Proof. Suppose that $(q, f) \vDash \Box_e\varphi$. Thus, $(q * e, g) \vDash \varphi$ for each $g \in NE(G, q * e)$ by Definition 3. Hence, $(q, g) \vDash \varphi$ for each $g \in NE(G, q)$ due to the equality $q * e = q$. In particular, $(q, f) \vDash \varphi$. ⊠

Lemma 2 *If* $(q,f) \vDash \Box_{p*r}\varphi$*, then* $(q,f) \vDash \Box_p\Box_r\varphi$*.*

Proof. Consider any $g \in NE(G, q*p)$. By Definition 3, it suffices to show that $(q*p, g) \vDash \Box_r\varphi$. Indeed, consider any $h \in NE(G,(q*p)*r)$. Again by Definition 3, we need to show that $((q*p)*r, h) \vDash \varphi$. Due to the associativity of operation $*$, it suffices to prove that $(q*(p*r), h) \vDash \varphi$. The latter is true due to assumption $(q,f) \vDash \Box_{p*r}\varphi$ and Definition 3. ⊠

Lemma 3 *If* $(q,f) \nvDash \Box_{p*r}\varphi$*, then* $(q,f) \vDash \Box_p\neg\Box_r\varphi$*.*

Proof. Assumption $(q,f) \nvDash \Box_{p*r}\varphi$, by Definition 3, implies that there is $g \in NE(G, q*(p*r))$ such that $(q*(p*r), g) \nvDash \varphi$. Thus, $g \in NE(G,(q*p)*r)$ and $((q*p)*r), g) \nvDash \varphi$ due to the associativity of operation $*$.

Consider any $h \in NE(G, q*p)$. By Definition 3, it suffices to prove that $(q*p, h) \vDash \neg\Box_r\varphi$. The latter, by Definition 3, is true because $g \in NE(G,(q*p)*r)$ and $((q*p)*r), g) \nvDash \varphi$. ⊠

Lemma 4 *If* $(q,f) \vDash \Box_p(\varphi \to \psi)$ *and* $(q,f) \vDash \Box_p\varphi$*, then* $(q,f) \vDash \Box_p\psi$*.*

Proof. Consider any $g \in NE(G, q*p)$. By Definition 3, it suffices to show that $(q*p, g) \vDash \psi$. Indeed, by Definition 3, assumption $(q,f) \vDash \Box_p(\varphi \to \psi)$ implies that $(q*p, g) \vDash \varphi \to \psi$, and assumption $(q,f) \vDash \Box_p\varphi$ implies that $(q*p, g) \vDash \varphi$. Therefore, $(q*p, g) \vDash \psi$ again by Definition 3. ⊠

Lemma 5 *If* $(r,g) \vDash \varphi$ *for all* $r \in P$ *and all* $g \in NE(G,r)$*, then* $(q,f) \vDash \Box_p\varphi$*.*

Proof. Consider any $h \in NE(G, q*p)$. By Definition 3, it suffices to show that $(q*p, h) \vDash \varphi$, which is true due to the assumption of the lemma. ⊠

The next corollary states the soundness of our logical system with respect to the game semantics.

Corollary 1 *If* $\vdash_{\langle P,e,*\rangle} \varphi$*, then* $(q,f) \vDash_G \varphi$ *for each game* G *based on the policy group* $\langle P,e,*\rangle$*, each* $q \in P$*, and each* $f \in NE(G,q)$*.*

6 Completeness

In this section we prove the completeness of our logical system with respect to the game semantics $\vDash$ specified in Definition 3. This proof is achieved in two steps. First, we define a class of Kripke models for our system and an auxiliary Kripke semantics for our system. We denote this semantics by $\Vdash$. Then, we prove that the following three statements are equivalent for each policy group $\langle P,e,*\rangle$ and each $\varphi \in \Phi(P)$:

(i) $\vdash \varphi$,

(ii) $(q, f) \vDash \varphi$, for each game G based on the policy group $\langle P, e, * \rangle$, each $p \in P$, and each Nash equilibrium $f \in NE(G, p)$,

(iii) $w \Vdash \varphi$ for each world w of each Kripke model based on the policy group $\langle P, e, * \rangle$.

The equivalence of these three statements is established by proving three implications:

(i) $\Rightarrow$ (ii) The first of these statements implies the second statement due to the soundness of our logical system with respect to the game semantics, which has been shown in Section 5.

(ii) $\Rightarrow$ (iii) In Section 6.3, we prove, by contrapositive, that the second statement implies the third via constructing a strategic game based on any given Kripke model. See Lemma 14 for details.

(iii) $\Rightarrow$ (i) Theorem 1 in Section 6.2 shows that the third statement implies the first statement.

Combined together, these three results prove the equivalence of statements (i), (ii), and (iii) above. We explicitly state the completeness result for the game semantics in Section 6.4 as Theorem 2.

6.1 Kripke Semantics

In this section we introduce an auxiliary Kripke semantics for our logical system that will eventually be used to prove the completeness theorem with respect to the original game semantics.

Our model could be viewed as an extension of a standard S5 Kripke model with an additional relation $\rightsquigarrow_p$ for each policy $p \in P$. The equivalence relation $\sim$ partitions set W into equivalence classes. The accessibility relation $\rightsquigarrow_p$ is a relation between these classes which is assumed to satisfy special forms of reflexivity, symmetry, transitivity, and functionality properties defined below. Similar to S5 Kripke models, our model also includes function π that specifies the set of worlds $\pi(v)$ in which a variable $v \in V$ is satisfied.

Definition 4 *For any policy group $\langle P, e, * \rangle$, a tuple $(W, \sim, \{\rightsquigarrow_p\}_{p \in P}, \pi)$ is called a Kripke model based on this policy group if the follow conditions are satisfied:*

1. *W is an arbitrary set of "worlds".*

2. *$\sim$ is an equivalence relation on W. The equivalence class of an arbitrary world $w \in W$ with respect to this relation is denoted by $[w]$.*

3. *$\rightsquigarrow_p$ is a binary relation on equivalence classes in $W/\sim$ that satisfies the following conditions for every $p, q \in P$ and every $w_1, w_2, w_3 \in W$:*

   (a) *Reflexivity: $[w_1] \rightsquigarrow_e [w_1]$,*

(b) *Symmetry: if* $[w_1] \rightsquigarrow_p [w_2]$, *then* $[w_2] \rightsquigarrow_{p^{-1}} [w_1]$,

(c) *Transitivity: if* $[w_1] \rightsquigarrow_p [w_2]$ *and* $[w_2] \rightsquigarrow_q [w_3]$, *then* $[w_1] \rightsquigarrow_{p*q} [w_3]$,

(d) *Functionality: if* $[w_1] \rightsquigarrow_p [w_2]$ *and* $[w_1] \rightsquigarrow_p [w_3]$, *then* $[w_2] = [w_3]$.

4. π *is an arbitrary function that maps propositional variables from* V *into subsets of* W.

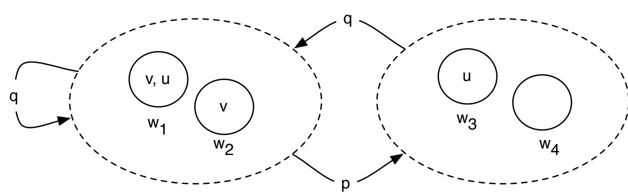

Figure 1: A Kripke model.

An example of a Kripke model is depicted in Figure 1. This model has four worlds: w_1, w_2, w_3, and w_4, partitioned into two equivalence classes with respect to the relation $\sim$. The accessibility relation $\rightsquigarrow$ is represented by arrows on this diagram. In this example, $\pi(v) = \{w_1, w_2\}$ and $\pi(u) = \{w_1, w_3\}$.

Next we define a Kripke semantics for our logical system by specifying the satisfiability relation $w \Vdash \varphi$. Recall that we use symbol $\Vdash$ for the satisfiability relation under the Kripke semantics and symbol $\vDash$ for the satisfiability relation under the game semantics.

Definition 5 *For any* $w \in W$ *and any* $\varphi \in \Phi(P)$, *the satisfiability relation* $w \Vdash \varphi$ *is defined recursively as follows:*

1. $w \Vdash v$ *if* $w \in \pi(v)$, *where* $v \in V$,

2. $w \Vdash \neg\varphi$ *if* $w \nVdash \varphi$,

3. $w \Vdash \varphi \to \psi$ *if* $w \nVdash \varphi$ *or* $w \Vdash \psi$,

4. $w \Vdash \Box_p\varphi$ *if* $w' \Vdash \varphi$ *for every* $w' \in W$ *such that* $[w] \rightsquigarrow_p [w']$.

For example, $w_1 \Vdash \Box_{p*q} v$ and $w_1 \Vdash \Box_{p*p}\bot$ for the Kripke model depicted in Figure 1.

The soundness of our logical system under the Kripke semantics is implication (i) $\Rightarrow$ (iii), for statements (i) and (iii) that were defined at the beginning of Section 6. We do not need to prove this result separately since it follows from results (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) that we show later in this section.

6.2 Completeness for Kripke Semantics

In this section we prove the completeness of our logical system with respect to the Kripke semantics. The proof follows the general scheme of the completeness proofs for modal logics with an addition of Definition 8 that specifies relation $\rightsquigarrow_p$ in the canonical model.

We now define the canonical Kripke model $(W, \sim, \{\rightsquigarrow_p\}_{p\in P}, \pi)$ based on an arbitrary policy group $\langle P, e, *\rangle$.

Definition 6 *Set W consists of all maximal consistent subsets of $\Phi(P)$.*

Definition 7 *$w_1 \sim w_2$ if the following condition is satisfied: $\Box_p\varphi \in w_1$ iff $\Box_p\varphi \in w_2$ for each $p \in P$ and each $\varphi \in \Phi(P)$.*

Corollary 2 *$\sim$ is an equivalence relation on W.* $\boxtimes$

Definition 8 *$[w_1]\rightsquigarrow_p[w_2]$ if the following condition is satisfied: $\Box_{p*q}\varphi \in w_1$ iff $\Box_q\varphi \in w_2$ for each $q \in P$ and each $\varphi \in \Phi(P)$.*

Lemma 6 *Relation $[w_1]\rightsquigarrow_p[w_2]$ is well-defined.*

Proof. Supposed that (i) statements $\Box_{p*q}\varphi \in w_1$ and $\Box_q\varphi \in w_2$ are either both true or both false, and (ii) $w_1 \sim w_1'$ and $w_2 \sim w_2'$. It is sufficient to show that statements $\Box_{p*q}\varphi \in w_1'$ and $\Box_q\varphi \in w_2'$ are either both true or both false. Indeed, by assumption $w_1 \sim w_1'$, statement $\Box_{p*q}\varphi \in w_1$ is equivalent to statement $\Box_{p*q}\varphi \in w_1'$. At the same time, by assumption $w_2 \sim w_2'$, statement $\Box_q\varphi \in w_1$ is equivalent to statement $\Box_q\varphi \in w_1'$. $\boxtimes$

The following lemmas verify reflexivity, symmetry, transitivity, and functionality properties from Definition 4 for the canonical Kripke model. In statements of these lemmas, we assume that $w_1, w_2, w_3 \in W$ and $p, q \in P$.

Lemma 7 $[w_1]\rightsquigarrow_e[w_1]$.

Proof. Note that $e * q = q$ due to the identity property of element e. Thus, $\Box_{e*q}\varphi \in w_1$ if and only if $\Box_q\varphi \in w_1$. $\boxtimes$

Lemma 8 *If $[w_1]\rightsquigarrow_p[w_2]$, then $[w_2]\rightsquigarrow_{p^{-1}}[w_1]$.*

Proof. We need to show that $\Box_{p^{-1}*q}\varphi \in w_2$ if and only if $\Box_q\varphi \in w_1$. Note that $q = e * q = (p * p^{-1}) * q = p * (p^{-1} * q)$. Thus, statement $\Box_q\varphi \in w_1$ is equivalent to statement $\Box_{p*(p^{-1}*q)}\varphi \in w_1$. The latter, by assumption $[w_1]\rightsquigarrow_p[w_2]$ and Definition 8, is equivalent to $\Box_{p^{-1}*q}\varphi \in w_2$. $\boxtimes$

Lemma 9 *If $[w_1]\rightsquigarrow_p[w_2]$ and $[w_2]\rightsquigarrow_q[w_3]$, then $[w_1]\rightsquigarrow_{p*q}[w_3]$.*

Proof. We need to show that $\Box_{p*q*r}\varphi \in w_1$ if and only if $\Box_r\varphi \in w_3$. Indeed, by assumption $[w_1]\rightsquigarrow_p[w_2]$, statement $\Box_{p*q*r}\varphi \in w_1$ is equivalent to $\Box_{q*r}\varphi \in w_2$. The last statement is equivalent to $\Box_r\varphi \in w_3$ by assumption $[w_2]\rightsquigarrow_q[w_3]$. $\boxtimes$

Lemma 10 *If* $[w_1]\rightsquigarrow_p[w_2]$ *and* $[w_1]\rightsquigarrow_p[w_3]$, *then* $[w_2] = [w_3]$.

Proof. Assume that $[w_1]\rightsquigarrow_p[w_2]$ and $[w_1]\rightsquigarrow_p[w_3]$. Without loss of generality, it is sufficient to show that if $\Box_q\varphi \in w_2$, then $\Box_q\varphi \in w_3$ for each $q \in P$ and each $\varphi \in \Phi(P)$. Suppose that $\Box_q\varphi \in w_2$. Then it follows from assumption $[w_1]\rightsquigarrow_p[w_2]$ and Definition 8 that $\Box_{p*q}\varphi \in w_1$. Therefore, $\Box_q\varphi \in w_3$, by assumption $[w_1]\rightsquigarrow_p[w_3]$ and Definition 8. $\boxtimes$

Definition 9 $\pi(v) = \{w \in W \mid v \in w\}$, *for all* $v \in V$.

This completes the definition of the canonical Kripke model $(W, \sim, \{\rightsquigarrow_p\}_{p\in P}, \pi)$.

Lemma 11 *For any* $p \in P$ *and any* $\varphi \in \Phi(P)$, *if* $\Box_p\varphi \notin w_1$, *then there is* $w_2 \in W$ *such that* $\varphi \notin w_2$ *and* $[w_1]\rightsquigarrow_p[w_2]$.

Proof. We first show that set

$$X_0 = \{\neg\varphi\} \cup \{\Box_q\psi \mid \Box_{p*q}\psi \in w_1\} \cup \{\neg\Box_r\chi \mid \neg\Box_{p*r}\chi \in w_1\}$$

is consistent. Assume the opposite. Then there are formulas $\Box_{p*q_1}\psi_1, \dots, \Box_{p*q_n}\psi_n$ and $\neg\Box_{p*r_1}\chi_1, \dots, \neg\Box_{p*r_m}\chi_m$ in w_1 such that

$$\Box_{q_1}\psi_1, \dots, \Box_{q_n}\psi_n, \neg\Box_{r_1}\chi_1, \dots, \neg\Box_{r_m}\chi_m \vdash \varphi.$$

Hence, by the Deduction theorem for propositional logic,

$$\vdash \Box_{q_1}\psi_1 \to (\dots \to (\Box_{q_n}\psi_n \to (\neg\Box_{r_1}\chi_1 \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)))\dots).$$

By the Necessitation inference rule,

$$\vdash \Box_p(\Box_{q_1}\psi_1 \to (\dots \to (\Box_{q_n}\psi_n \to (\neg\Box_{r_1}\chi_1 \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)))\dots)).$$

By axiom (A4) and the Modus Ponens inference rule,

$$\begin{aligned}\Box_p\Box_{q_1}\psi_1 \vdash \quad & \Box_p(\Box_{q_2}\psi_2 \to (\dots \to (\Box_{q_n}\psi_n \to (\neg\Box_{r_1}\chi_1 \\ & \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)))\dots)).\end{aligned}$$

By axiom (A2),

$$\begin{aligned}\Box_{p*q_1}\psi_1 \vdash \quad & \Box_p(\Box_{q_2}\psi_2 \to (\dots \to (\Box_{q_n}\psi_n \to (\neg\Box_{r_1}\chi_1 \\ & \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)))\dots)).\end{aligned}$$

By repeating the last two steps $(n-1)$ times,

$$\Box_{p*q_1}\psi_1, \dots, \Box_{p*q_n}\psi_n \vdash \Box_p(\neg\Box_{r_1}\chi_1 \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)).$$

By axiom (A4) and the Modus Ponens inference rule,

$$\Box_{p*q_1}\psi_1, \dots, \Box_{p*q_n}\psi_n, \Box_p\neg\Box_{r_1}\chi_1$$
$$\vdash \Box_p(\neg\Box_{r_2}\chi_2 \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)).$$

By axiom (A3),

$$\Box_{p*q_1}\psi_1, \dots, \Box_{p*q_n}\psi_n, \neg\Box_{p*r_1}\chi_1$$
$$\vdash \Box_p(\neg\Box_{r_2}\chi_2 \to (\dots \to (\neg\Box_{r_m}\chi_m \to \varphi)\dots)).$$

By repeating the last two steps $(m-1)$ times,

$$\Box_{p*q_1}\psi_1, \dots, \Box_{p*q_n}\psi_n, \neg\Box_{p*r_1}\chi_1, \dots, \neg\Box_{p*r_m}\chi_m \vdash \Box_p\varphi.$$

Recall that formulas $\Box_{p*q_1}\psi_1, \dots, \Box_{p*q_n}\psi_n$ and $\neg\Box_{p*r_1}\chi_1, \dots, \neg\Box_{p*r_m}\chi_m$ are in w_1. Thus, $w_1 \vdash \Box_p\varphi$. Hence, $\Box_p\varphi \in w_1$ due to the maximality of set w_1, which contradicts to the assumption of the lemma that $\Box_p\varphi \notin w_1$. Therefore, set X_0 is consistent. Let w_2 be a maximal consistent extension of X_0. Note that $\varphi \notin w_2$ because $\neg\varphi \in X_0 \subseteq w_2$ and set w_2 is consistent. By the choice of X_0, we have $[w_1] \rightsquigarrow_p [w_2]$. ⊠

Lemma 12 *$w \Vdash \varphi$ if and only if $\varphi \in w$, for each formula $\varphi \in \Phi(P)$ and each world $w \in W$ of the canonical Kripke model.*

Proof. We prove the lemma by induction on the structural complexity of formula φ.

Case I: If formula φ is a propositional variable, then the required follows from Definition 9 and Definition 5.

Case II: If formula φ is a negation or an implication, then the required follows, in the standard way, from the induction hypothesis and Definition 5 due to the maximality and consistency of set w.

Case III: If formula φ is of the form $\Box_p\psi$ for some $p \in P$ and some $\psi \in \Phi(P)$.
$(\Rightarrow)$: Suppose that $\Box_p\psi \notin w$. Thus, by Lemma 11, there is $w' \in W$ such that $[w] \rightsquigarrow_p [w']$ and $\psi \notin w'$. Hence, by the induction hypothesis, $w' \nVdash \psi$. Therefore, $w \nVdash \Box_p\psi$ by Definition 5.

$(\Leftarrow)$: Assume that $w \nVdash \Box_p\psi$. Thus, by Definition 5, there exists $w' \in W$ such that $[w] \rightsquigarrow_p [w']$ and $w' \nVdash \psi$. Hence, by the induction hypothesis, $\psi \notin w'$. Then $\Box_e\psi \notin w'$ due to axiom (A1) and the maximality of set w'. Thus, $\Box_{p*e}\psi \notin w$ by Definition 8. Therefore, $\Box_p\psi \notin w$ since $p * e = p$. ⊠

We are now ready to state and prove the completeness theorem for our logical system with respect to the Kripke semantics.

Theorem 1 *For any $\varphi \in \Phi(P)$, if $w \Vdash \varphi$ for each world $w \in W$ of each Kripke model $(W, \sim, \{\rightsquigarrow_p\}_{p\in P}, \pi)$ based on policy group $\langle P, e, * \rangle$, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Let w be any maximal consistent subset of $\Phi(P)$ such that $\neg\varphi \in w$. Then, $w \nVdash \varphi$ by Lemma 12. $\boxtimes$

6.3 Canonical Game

In this section, for a given Kripke model $\mathcal{K} = (W, \sim, \{\rightsquigarrow_p\}_{p\in P}, \pi)$ based on the policy group $\langle P, e, *\rangle$, we define a *canonical* multi-policy game. Our goal is to define the game in such a way that an arbitrary formula is true in an arbitrary world if and only if this formula is true in a "corresponding" Nash equilibrium of the game. This "correspondence" is formally specified in Lemma 14.

The canonical game $G^w_{\mathcal{K}}$ is defined for each world $w \in W$. The game has three players. The first player has the set of all worlds W as her strategies. Under the policy p, she is rewarded to choose a strategy $s_1 \in W$ such that $[w]\rightsquigarrow_p[s_1]$ is true in the Kripke model $\mathcal{K}$. If such a world s_1 does not exist, the two other players are paid to play the matching pennies game. In this case the canonical game has no Nash equilibria under the policy p. Now we define the canonical game $G^w_{\mathcal{K}} = \langle N, \{A_i\}_{i\in N}, \{u^p_i\}^{p\in P}_{i\in N}, \ell\rangle$ formally.

Definition 10 *The set of players N consists of three players: 1, 2, and 3.*

Definition 11 *The set of strategies A_i available to a player $i \in N$ is defined as follows:*

1. $A_1 = W$,
2. $A_2 = A_3 = \{head, tail\}$.

The next three definitions specify the utility functions for the players.

Definition 12

$$u^p_1(\langle s_1, s_2, s_3\rangle) = \begin{cases} 1, & \textit{if } [w]\rightsquigarrow_p[s_1],\\ 0, & \textit{otherwise.}\end{cases}$$

Definition 13

$$u^p_2(\langle s_1, s_2, s_3\rangle) = \begin{cases} 1, & \textit{if } s_2 = s_3 \textit{ or there is } w' \in W \textit{ such that } [w]\rightsquigarrow_p[w'],\\ -1, & \textit{otherwise.}\end{cases}$$

Definition 14

$$u^p_3(\langle s_1, s_2, s_3\rangle) = \begin{cases} 1, & \textit{if } s_2 \neq s_3 \textit{ or there is } w' \in W \textit{ such that } [w]\rightsquigarrow_p[w'],\\ -1, & \textit{otherwise.}\end{cases}$$

Our intention is for a propositional variable to be satisfied in a Nash equilibrium $\langle s_1, s_2, s_3\rangle$ of the canonical game if and only if it is satisfied in the world s_1 of the Kripke model $\mathcal{K}$. Thus, we explicitly define the satisfiability of propositional variables in strategy profiles accordingly.

Definition 15 *$\ell(v) = \{\langle s_1, s_2, s_3\rangle \mid s_1 \in \pi(v)\}$, for each $v \in V$.*

This concludes the definition of the canonical game $G^w_{\mathcal{K}}$. The next lemma describes the set of Nash equilibria of the canonical game.

Lemma 13 *$NE(G^w_{\mathcal{K}}, p) = \{\langle s_1, s_2, s_3\rangle \in A_1 \times A_2 \times A_3 \mid [w] \rightsquigarrow_p [s_1]\}$.*

Proof. First, assume that $[w] \rightsquigarrow_p [s_1]$. In this case, according to Definition 12, Definition 13, and Definition 14, all three players have the maximal possible payoff 1. Hence, $\langle s_1, s_2, s_3\rangle \in NE(G^w_{\mathcal{K}}, p)$.

Next, suppose that it is not true that $[w] \rightsquigarrow_p [s_1]$. There are two cases to consider.

Case I: If there exists $w' \in W$ such that $[w] \rightsquigarrow_p [w']$, then $\langle s_1, s_2, s_3\rangle$ is not a Nash equilibrium because player 1 can switch strategy from s_1 to w' to increase her payoff.

Case II: If there is no $w' \in W$ such that $[w] \rightsquigarrow_p [w']$, then players 2 and 3 are playing the matching pennies game. Since the matching pennies game has no pure Nash equilibria, $\langle s_1, s_2, s_3\rangle \notin NE(G^w_{\mathcal{K}}, p)$. $\boxtimes$

The next lemma connects the satisfiability relation for the Kripke model $\mathcal{K}$ and the satisfiability relation for the canonical game $G^w_{\mathcal{K}}$.

Lemma 14 *$(p, \langle s_1, s_2, s_3\rangle) \vDash \psi$ if and only if $s_1 \Vdash \psi$, for each $\psi \in \Phi(P)$, each $p \in P$ and each $\langle s_1, s_2, s_3\rangle \in NE(G^w_{\mathcal{K}}, p)$.*

Proof. We prove the lemma by induction on the structural complexity of formula ψ.

Case I: Assume that ψ is a propositional variable $v \in V$. Statement $(p, \langle s_1, s_2, s_3\rangle) \vDash v$ is equivalent to $\langle s_1, s_2, s_3\rangle \in \ell(v)$, by Definition 3. The latter, by Definition 15, is equivalent to $s_1 \in \pi(v)$, which, by Definition 5, is equivalent to $s_1 \Vdash v$.

Case II: If formula ψ is a negation or an implication, then the required follows from Definition 5, Definition 3, and the induction hypothesis.

Case III: Suppose that formula ψ has form $\Box_q \chi$.

$(\Leftarrow)$: Assume that $s_1 \nVdash \Box_q \chi$. By Definition 5, there is a world $u \in W$ such that $u \nVdash \chi$ and $[s_1] \rightsquigarrow_q [u]$. By the assumption of the lemma, $\langle s_1, s_2, s_3\rangle \in NE(G^w_{\mathcal{K}}, p)$. Thus, $[w] \rightsquigarrow_p [s_1]$ by Lemma 13. By the Transitivity condition from Definition 4, we have $[w] \rightsquigarrow_{p*q} [u]$. Then, by Lemma 13, $\langle u, 1, 1\rangle \in NE(G^w_{\mathcal{K}}, p*q)$. By the induction hypothesis, statement $u \nVdash \chi$ implies that $(p*q, \langle u, 1, 1\rangle) \nvDash \chi$. Therefore, by Definition 3, $(p, \langle s_1, s_2, s_3\rangle) \nVdash \Box_q \chi$.

$(\Rightarrow)$: Assume that $(p, \langle s_1, s_2, s_3\rangle) \nvDash \Box_q \chi$. Hence, by Definition 3, there exists $\langle s'_1, s'_2, s'_3\rangle \in NE(G^w_{\mathcal{K}}, p*q)$ such that $(p*q, \langle s'_1, s'_2, s'_3\rangle) \nvDash \chi$. Note that by Lemma 13, $[w] \rightsquigarrow_{p*q} [s'_1]$. Hence, by the induction hypothesis, $s'_1 \nVdash \chi$.

By the assumption of the lemma, $\langle s_1, s_2, s_3\rangle \in NE(G^w_{\mathcal{K}}, p)$. Then, $[w] \rightsquigarrow_p [s_1]$ by Lemma 13. By the Symmetry condition from Definition 4, we have $[s_1] \rightsquigarrow_{p^{-1}} [w]$.

Thus, by the Transitivity condition from Definition 4, $[s_1] \rightsquigarrow_{p^{-1}*(p*q)} [s'_1]$. By basic properties of group operation $*$, we have $p^{-1}*(p*q) = (p^{-1}*p)*q = e*q = q$. Hence, $[s_1] \rightsquigarrow_q [s'_1]$. Therefore, $s_1 \nVdash \Box_q \chi$ by Definition 5. $\boxtimes$

6.4 Completeness: Final Step

In this section we formally state and prove the completeness theorem for our system with respect to the game semantics.

Theorem 2 *For any policy group $\langle P, e, * \rangle$ and any $\varphi \in \Phi(P)$, if $(q, f) \vDash \varphi$ for each multi-policy game $G = (N, \{A_i\}_{i \in N}, \{u_i^p\}_{i \in N}^{p \in P}, \ell)$ and each $f \in NE(G, p)$, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Thus, by Theorem 1, there is a world $w \in W$ of a Kripke model $\mathcal{K} = (W, \sim, \{\rightsquigarrow_p\}_{p \in P}, \pi)$ with the policy group $\langle P, e, * \rangle$ such that $w \nVdash \varphi$. Consider the canonical game $G^w_{\mathcal{K}}$. Note that $[w] \rightsquigarrow_e [w]$ due to the Reflexivity condition in Definition 4. Hence, $\langle w, 1, 1 \rangle \in NE(G^w_{\mathcal{K}}, e)$, by Lemma 13. Therefore, $(e, \langle w, 1, 1 \rangle) \nvDash \varphi$ by Lemma 14. $\boxtimes$

7 Discussion

In this article we introduced a logical system for reasoning about economic policies and proved the completeness of this system with respect to a strategic game semantics. So far, we have only considered policies that can be expressed as adjustments to utility functions in strategic games. In conclusion, we would like to briefly discuss how our approach could be adapted to some of other types of economic policies.

One of the monetary tools commonly employed by central banks to regulate economy is buying and selling reserve assets (such as bonds). This policy could be represented in a similar setting by considering the government as a special player in the game. Enforcing a policy in this setting corresponds to the government committing to a particular strategy in the game.

The other common monetary tool consists of regulating how much money banks must keep in reserve and, thus, not make available for loans. This type of policies could be represented as a change to the sets of strategies accessible to the players.

Both extensions described above would require a non-trivial modification in the syntax and game semantics of our system. The completeness theorem for either extension remains an open problem.

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