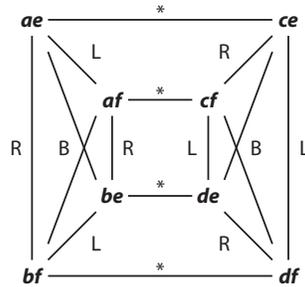


# A Cube of Isotopies

John Conway and Derek Smith

Several years ago, John and I were working on a second edition of *On Quaternions and Octonions* when I hit several months of a heavy teaching schedule and could not visit Princeton regularly. By the time the semester ended, John's health had declined, and we waited to return to the project. Sadly, the time for us to work together had passed.

Here is an intriguing result that was part of our investigation of various multiplication doubling formulae that extend the 8-dimensional octonions to 16-dimensional algebras. The eight vertices of the cube correspond (with different names) to the eight "pre-eminent formulae" studied by Warren D. Smith in "Quaternions, octonions, and now, 16-ons and  $2^n$ -ons." We found that the multiplications are related by conjugation ( $\star$ ) and isotopies:



If  $\circ$  and  $\star$  are two multiplications joined by an edge labeled  $B$ ,  $L$ , or  $R$ , they are related by the respective involutory Moufang maps

$$\begin{aligned} B(x \circ y) &= L(x) \star R(y) \\ -L(x \circ y) &= B(x) \star L(y) \\ -R(x \circ y) &= R(x) \star B(y), \end{aligned}$$

where  $B(z) = izi$ ,  $L(z) = iz$ , and  $R(z) = zi$ , and  $i$  is the imaginary unit extending the octonions to 16 dimensions.

I look forward to preparing and sharing our work on this topic in the near future.

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