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Partitions of $AG(4, 3)$ into maximal caps

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ABSTRACT

In a geometry, a maximal cap is a collection of points of largest size no three of which are collinear. In $AG(4, 3)$, maximal caps contain 20 points; the 81 points of $AG(4, 3)$ can be partitioned into 4 mutually disjoint maximal caps together with a single point P, where every pair of points that makes a line with P lies entirely inside one of those caps. The caps in a partition can be paired up so that both pairs are either in exactly one partition or they are both in two different partitions. This difference determines the two equivalence classes of partitions of $AG(4, 3)$ under the action by affine transformations.

1. Introduction

A *k-cap* (or briefly a *cap*) in $AG(n, 3)$ is a set of k points containing no three points on a line; a *maximal cap* is a cap of the largest possible size. A cap is *complete* if it is not a subset of a larger cap. There are caps in $AG(n, 3)$, $n \geq 3$, which are complete but smaller than a maximal cap.

The elements of $AG(n, 3)$ can be written as n-tuples with coordinates in $\mathbb{Z}_3$, so the full transformation group of $AG(n, 3)$ is the affine group $Aff(n, 3) = GL(n, 3) \ltimes \mathbb{Z}_3^n$, where $(A, \vec{b})$ represents the transformation $\vec{v} \mapsto A\vec{v} + \vec{b}$. Alternatively, a transformation permutes the points of $AG(n, 3)$ by mapping $n + 1$ affinely independent set of points to any $n + 1$ affinely independent points. The applet Swingset, developed by Coleman, Hartshorn, Long and Mills [4] provides a nice way to visualize these affine transformations using the card game SET® [16]. Caps are invariant under the action of $Aff(n, 3)$.

There is a large body of work by many authors examining the maximal (and complete) caps in $PG(n, 3)$ and $AG(n, 3)$. The maximal caps in the geometries $PG(4, 3)$ and $AG(4, 3)$ were first enumerated in 1970, in a paper (written in Italian) by G. Pellegrino [14]. In 1983, R. Hill [10] proved that all maximal caps in $AG(4, 3)$ are affinely equivalent. Aided by the visualization provided by SET®, a rich geometric structure to these caps has been discovered. A. Forbes [8] found that the 81 points in $AG(4, 3)$ can be partitioned into 4 mutually disjoint maximal caps with a single point $\vec{a}$ left. G. Gordon [9] realized that any pair of points that make a line with $\vec{a}$ (there are 40 such pairs) lie in one of the caps in the partition. It is the goal of this paper to explore more of the structure of these partitions.

The notion of partitioning finite affine or projective spaces into caps has been studied as well. There is a good summary from 2003 in Section 13, "Large sets of caps", in Bierbrauer's survey on caps [2]. However, in most cases, authors were not looking exclusively at maximal caps. In [11], B. Kestenband looked at partitions of $PG(2n, q^2)$ into disjoint caps, although those caps are not necessarily maximal (or even complete). Then, in [12], he extended that to $AG(n, q^2)$, finding a partition of that space into "affine caps", which are unions of disjoint affine subgeometries so that no three points are collinear unless they contained in the same affine subgeometry. Ebert [6] extended these results to a partition of $PG(2n - 1, q)$ into caps of

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Table 1
All known sizes of maximal caps in $AG(n, 3)$.

$AG(1, 3)$	$AG(2, 3)$	$AG(3, 3)$	$AG(4, 3)$	$AG(5, 3)$	$AG(6, 3)$
2	4	9	20	45	112

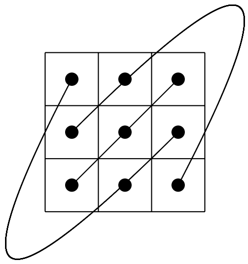


Fig. 1. $AG(2, 3)$ with one set of diagonal lines shown.

size $q^2 + 1$ (n even, $n \geq 2$), which gives rise to a partition of $PG(3, 3)$ into four caps of size 10. Because $AG(3, 3)$ is obtained by deleting a plane from $PG(3, 3)$, which could contain at most 4 points of any cap, Ebert's partition will give rise to a partition of $AG(3, 3)$ into four caps. In Section 2, we give a partition of $AG(3, 3)$ into three disjoint maximal caps. Theorem 22 in Bierbrauer's paper states that if $PG(k, 3)$ can be partitioned into $2l$ caps, then $AG(k + 2, 3)$ can be partitioned into $3l$ caps. This can be used to give a partition of $AG(4, 3)$ into 6 caps. We improve the partition to 5 caps, 4 of which are maximal.

The symmetry group of $AG(n, 3)$ acts transitively on maximal caps as sets; here, we show that the action is not 2-transitive on disjoint caps. Further, we show that the symmetry group does not act transitively on partitions of the affine geometry into maximal caps, and we find the equivalence classes of the action. We will look at partitions of $AG(n, 3)$, for $n = 2$, 3 and 4; we show in Section 2 that the action of the symmetry group is transitive on partitions for $AG(2, 3)$ and $AG(3, 3)$. In Section 3, we show that the partitions in $AG(4, 3)$ are in two affine equivalence classes in $AG(4, 3)$ and isolate the fundamental difference between those classes.

Finally, $Aff(4, 3)$ is of order 1,965,150,720. In Section 4, we briefly examine various subgroups of this group that fix particular caps and partitions as sets.

2. Caps in $AG(n, 3)$, with a focus on $n < 4$

Table 1 enumerates the known sizes of maximal caps in $AG(n, 3)$ for $n \leq 6$, the only sizes known at this time.

The sizes for maximal caps in dimensions 1 and 2 can be found by inspection. The size for dimension 3 was first analyzed by Bose in 1947 [3]. In 1970, Pellegrino provided the first proof that there are 20 points in a maximal cap in $AG(4, 3)$ [14] by finding caps in $PG(4, 3)$, one of which lies entirely in an induced $AG(4, 3)$. Edel, Ferret, Landjev and Storme first classified the maximal caps in $AG(5, 3)$ in 2002 [7]. The results in dimensions 4 and 5 came from looking at caps in the projective space $PG(n, 3)$, and removing points. In 2008, A. Potechin found the size of the maximal caps in $AG(6, 3)$ [15]; this result came from looking at caps in $AG(5, 3)$ and analyzing how those can extend to the higher dimension. It is still not known how large maximal caps are in dimensions larger than 6. In all dimensions where the sizes of maximal caps are known, all maximal caps are affinely equivalent. Hill first showed this for $AG(4, 3)$ in 1983 [10]; this result has been extended to all dimensions where the sizes of maximal caps are known in the papers that first identified the maximal caps.

The points in $AG(n, 3)$ can be realized as n-tuples of elements of $\mathbb{F}_3$. In this case, as Davis and Maclagan [5] point out, lines are easy to identify: three points in $AG(n, 3)$ are collinear if and only if their sum is $\vec{0}$ mod 3.

In dimensions 2 and 4, the maximal caps consist of pairs of points from a pencil of lines through a fixed point; we call that point the *anchor point*. In dimensions 3 and 5, maximal caps sum to $\vec{0}$ mod 3. In dimension 3 and lower, we can find results about caps simply by inspection. In this paper, we extend this direct analysis to dimension 4. We begin in dimension 2.

To aid in the visualization, we will use the same scheme as is used by Davis and Maclagan [5]: $AG(2, 3)$ is represented by a 3×3 grid, as pictured below in Fig. 1. There are 12 lines in $AG(2, 3)$: 3 horizontal, 3 vertical, 3 diagonals as pictured, and the 3 diagonals in the opposite direction.

Proposition 2.1. *In* $AG(2, 3)$, *a maximal cap has* 4 *points and consists of two lines through an anchor point, with the anchor removed; all maximal caps are affinely equivalent.* $AG(2, 3)$ *can be partitioned into two disjoint caps together with their common anchor point. Any two partitions are affinely equivalent.*

Proof. The structure of the maximal caps and their affine equivalence can easily be determined by inspection. The four lines through the point $\vec{0}$ (in the upper left) are shown in Fig. 2 as a pair of points of the same size and shading. Any two of these pairs will form a maximal cap. The anchor is uniquely determined by the cap, since the sum of the coordinates for the points

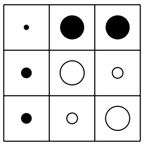


Fig. 2. A partition of $AG(2, 3)$ into 4-caps with anchor point in the upper left.

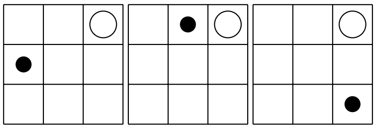


Fig. 3. $AG(3, 3)$ with two sets of collinear points shown.

in a cap gives the coordinates for the anchor. Thus, since any point has four lines through it, the remaining four points must be a cap with the same anchor. An affine transformation is determined by the images of the anchor, one large black point and one small black point, so any two caps are equivalent; since the partition is determined by a cap, so all partitions are affinely equivalent as well. □

We will represent $AG(3, 3)$ by three 3×3 grids. Two examples of lines are shown in Fig. 3. When coordinatizing $AG(3, 3)$, we can have the first coordinate give the $AG(2, 3)$ subgrid, and the last two give the point within that subgrid. Thus, a line will either consist of three points in one subgrid of $AG(3, 3)$ in the same position as a line from $AG(2, 3)$ (so the first coordinates will be the same), or three points with one in each of the three subgrids such that, if you superimpose the three subgrids, the points are either in the same position (the open circles in Fig. 3: the last two coordinates will be the same) or are in the position as a line in $AG(2, 3)$ (the solid dots in Fig. 3).

A set of $q^2 + 1$ points with no 3 on a line is called *ovoid* in $PG(3, q)$; at each point P of the ovoid, all lines through P that meet the ovoid only at P lie in a single (tangent) plane. If that plane is deleted, we get $AG(3, q)$, with a cap of size q^2. So, for $AG(3, 3)$, a maximal cap has 9 points.

In $AG(3, 3)$, the caps sum to $\vec{0}$, and there is no anchor. $AG(3, 3)$ can be partitioned into 3 disjoint maximal caps. It is interesting to note that all maximal caps in $AG(5, 3)$ have 45 points, which also sum to $\vec{0}$; could it always be true that maximal caps sum to $\vec{0} \bmod 3$ in odd dimensions? However, $AG(5, 3)$ cannot have a similar decomposition into disjoint caps, as 45 does not divide 243.

Proposition 2.2. (1) *In $AG(3, 3)$, all maximal caps are affinely equivalent; the coordinates for a maximal cap sum to* $\vec{0}$ mod 3.
(2) *$AG(3, 3)$ can be partitioned into three mutually disjoint maximal caps. Every maximal cap in $AG(3, 3)$ is in a unique partition of $AG(3, 3)$; thus, all partitions are equivalent.*

Proof. (1) An example of a maximal cap in $AG(3, 3)$ is pictured in Fig. 4; label these points $\vec{b}_1, \ldots, \vec{b}_9$. The reader can verify that there are no lines in the set of points and that the sum of coordinates is $\vec{0}$ mod 3.

In 1947, R.C. Bose first proved that the size of a maximal cap in $PG(3, s)$ is $s^2 + 1$ when s is a power of an odd prime [3] and used a quadric surface to find a cap of that size. He showed that certain planes intersect the cap in a single point, so deleting that plane gives a cap in $AG(3, q)$ of size q^2. In 1955, Barlotti [1] and Panella [13] independently showed that, when q is odd, all maximal caps in $PG(3, q)$ are the $q^2 + 1$ points of an elliptic quadric, so all are projectively equivalent. This gives us the rest of (1).

(2) Fig. 5 shows a decomposition of $AG(3, 3)$ into 3 disjoint maximal caps. Notice that one of the caps is $\vec{b}_1, \ldots, \vec{b}_9$. Since all 9-caps of $AG(3, 3)$ are affinely equivalent, it suffices to show that this partition is the only partition containing $\vec{b}_1, \ldots, \vec{b}_9$.

Given any maximal cap C and 3 parallel planes of $AG(3, 3)$, C must intersect those 3 planes in sets of size 4, 4 and 1 or 3, 3 and 3. (The only other possibility would be for C to intersect those planes in 4, 3 and 2 points, since a cap cannot have more than 4 points in a plane. However, we can find an affine transformation so that each plane corresponds to one coordinate of the vectors; if the cap intersects those 3 planes in 4, 3 and 2 points, the sum of the cap for that coordinate cannot be 0 mod 3.) Thus, given $C = \{\vec{b}_1, \ldots, \vec{b}_9\}$, for any partition containing C, the other two caps must intersect the three planes in 1 or 4 points. In the first two planes, if another cap has a 4-point intersection with the plane, then viewing the plane as $AG(2, 3)$, the anchor for that cap must be the same as the anchor for the 4 points of C, so the only possibility for the first 2 planes is what is shown. Thus, the other two caps comprising the partition are completely determined. □

3. Disjoint and intersecting maximal caps of *AG*(4, 3)

We will represent $AG(4, 3)$ by a 9×9 grid, which we can view as three copies of $AG(3, 3)$ or nine copies of $AG(2, 3)$ (arranged as $AG(2, 3)$). A line will consist of three points that appear either in the same 3×3 subgrid (as a line in $AG(2, 3)$),

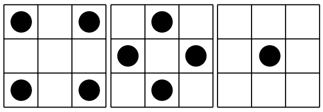


Fig. 4. A maximal 9-cap in $AG(3, 3)$.

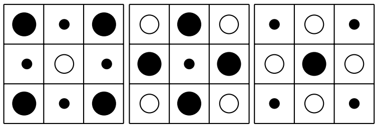


Fig. 5. $AG(3, 3)$ partitioned into 3 disjoint maximal caps.

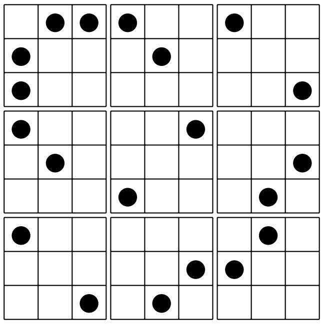


Fig. 6. A maximal cap S in $AG(4, 3)$; the anchor point is in the upper left.

or in three subgrids that correspond to a line in $AG(2, 3)$ so that, when the subgrids are superimposed, the points are either in the same position or they are a line in $AG(2, 3)$. When coordinatizing $AG(4, 3)$, we can have the first two coordinates give the $AG(2, 3)$ subgrid, and the second two give the point within that subgrid. A maximal cap in $AG(4, 3)$ contains 20 points in 10 pairs, where each pair completes a line with the anchor point; one such cap S is shown in Fig. 6. All maximal caps are affinely equivalent (Pellegrino [14] and Hill [10]). Considering the points as 4-tuples, S is the first cap lexicographically with $\vec{0}$ as its anchor point.

Lemma 3.1. *For any maximal cap S in $AG(4, 3)$, there exists an anchor point $\vec{a}$, so that the cap consists of 10 pairs of points, each of which forms a line with $\vec{a}$. The sum of the coordinates of the points in S is $-\vec{a} \bmod 3$, so the anchor point is unique.*

Proof. One can verify that the set S pictured in Fig. 6 contains no lines, so it must be a maximal cap. S consists of 10 pairs of points, where the third point completing the line for each pair is the point in the upper left, $\vec{0}$. Since any other maximal cap is affinely equivalent to S, the same must be true for all caps.

Further, suppose that S_1 is an arbitrary maximal cap with an anchor point $\vec{a}$. Since the coordinates of three collinear points sum to $\vec{0} \bmod 3$, if the coordinates for the points in S_1 are summed with $10\vec{a}$, the result must be $\vec{0} \bmod 3$. Thus, the sum of the points in S_1 is $-\vec{a} \bmod 3$. □

The following lemma was originally verified by Forbes via a computer search; we give a direct proof. Note that an affine transformation fixing the point corresponding to $\vec{0}$ is a linear transformation. This will simplify some of our arguments.

Lemma 3.2. *There are 8424 maximal caps with anchor $\vec{0}$; they are all linearly equivalent.*

Proof. All maximal caps in $AG(4, 3)$ are affinely equivalent. The elements of $GL(4, 3)$ send caps with anchor $\vec{0}$ to caps with anchor $\vec{0}$. Thus, by the Orbit–Stabilizer Theorem, we can count the number of caps with anchor $\vec{0}$ by counting the matrices in $GL(4, 3)$ that send the cap S to itself. Since a linear transformation is determined by its action on a basis, we will find such a basis among the vectors corresponding to points in S. If we order the points of S (as pictured in Fig. 6) lexicographically as $c_1, -c_1, c_2, -c_2, \ldots, c_{10}, -c_{10}$, we can see that c_1, c_2, c_3 and c_5 are linearly independent. So we will determine a matrix in $GL(4, 3)$ fixing S by specifying the images of those vectors.

Looking at Fig. 6, c_1 can be sent to any of the 20 points. c_2 can be sent to any of the 18 points that do not include the image of c_1 and $-c_1$. The image of c_3 is restricted by the fact that once it is chosen, all points from S in the hyperplane determined by c_1, c_2 and c_3 must go to points in S as well, since all points in that hyperplane are linear combinations of c_1, c_2 and c_3. Not all choices for the image of c_3 will work. Similarly, the image of c_5, the last point in S not in that hyperplane, does not have full freedom. The reader can verify that, once the image of c_1 and c_2 are chosen, there are only 8 possibilities for the images

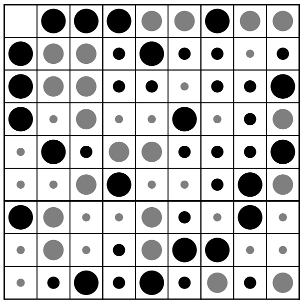


Fig. 7. A partition of $AG(4, 3)$ into 4 disjoint maximal caps; the anchor point $\vec{0}$ is in the upper left.

of c_3 and c_5. Thus, there are $20 \cdot 18 \cdot 8 = 2880$ matrices that fix S as a cap. Thus, there are $|GL(4, 3)|/2880 = 8424$ caps with anchor $\vec{0}$. □

The next theorem shows that $AG(4, 3)$ can be partitioned into 4 disjoint maximal caps together with their common anchor, just as $AG(2, 3)$ was. This fact was first noticed by Forbes [8] and Gordon [9].

Theorem 3.3. *$AG(4, 3)$ can be partitioned into 4 mutually disjoint maximal caps together with their common anchor $\vec{a}$.*

Proof. One such partition, where S pictured above is one of the maximal caps, is shown in Fig. 7. The reader can verify that the claims in the theorem hold for this partition. □

The goal of the rest of this paper is to study these partitions.

The next proposition has been verified by computer search. It would be instructive to have a geometric proof of this fact, as it implies something important about the structure of maximal caps. The proposition is very useful in understanding the structure of the partitions, for it shows that the partitions in Theorem 3.3 are the only kind of partitions of $AG(4, 3)$ that can include disjoint maximal caps.

Proposition 3.4. *Any two maximal caps with different anchor points intersect in at least one point.*

Proof. Because any two maximal caps are affinely equivalent, it suffices to verify that a given maximal cap has nonempty intersection with all caps with all other anchor points. Let S be the maximal cap with anchor $\vec{0}$ pictured in Fig. 6. Let $\{S_1, \ldots, S_{8424}\}$ be the set of 8424 maximal caps with anchor $\vec{0}$. For a cap S_i in that set, if we add $\vec{a}$(mod 3) to each point in S_i (which we write as $S_i + \vec{a}$), we get a cap with anchor $\vec{a}$. Thus, $\{S_i + \vec{a}\}$ is the set of 8424 maximal caps with anchor $\vec{a}$.

A computer check ran through all 80 possible anchor points $\vec{a}$ and verified that S and $S_i + \vec{a}$ had nonempty intersection for $1 \le i \le 8424$. □

The same computer check verified the first claim in the next proposition. The last claims in the proposition were shown by a different computer search by Forbes [8].

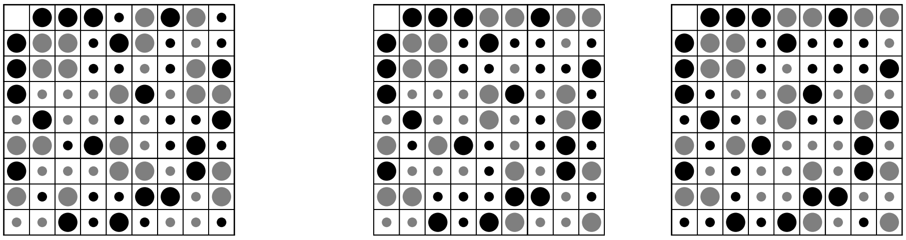
Proposition 3.5. *Let S be a maximal cap with anchor $\vec{0}$. There are 198 maximal caps (necessarily with anchor $\vec{0}$) disjoint from S. There are 216 different partitions of $AG(4, 3)$ containing S as a block; each of the 198 caps disjoint from S is in at least one of the 216 partitions.*

While the group $Aff(n, 3)$ acts transitively on maximal caps, there are three equivalence classes for pairs (S_1, S_2) of disjoint caps. Consider Figs. 8 and 9. Let S be the maximal cap with anchor $\vec{0}$ in large black dots (this is the same cap pictured in Fig. 6). Three different caps C disjoint from S are pictured in large gray dots in Figs. 8(a), (b) and 9. In each case, there are 40 points not in $\{\vec{0}\} \cup S \cup C$. In Fig. 8(a), those points can be partitioned into 2 disjoint maximal caps in only one way; in Fig. 8(b), they can be partitioned into 2 disjoint maximal caps in two different ways. In Fig. 9, they can be partitioned into 2 disjoint maximal caps in six different ways.

Let S be a maximal cap with anchor $\vec{0}$. By Proposition 3.5, any of the 198 maximal caps disjoint from S is in a partition. A computer check has verified that there are 36 maximal caps disjoint from S that appear in only one partition of $AG(4, 3)$ containing S; there are 90 maximal caps disjoint from S that appear in exactly two partitions of $AG(4, 3)$ containing S; and there are 72 maximal caps disjoint from S that appear in exactly six partitions of $AG(4, 3)$ containing S. This leads to the following definition.

Definition. Given a maximal cap S, let S' be a cap (necessarily with the same anchor point) disjoint from S. If $\{S, S'\}$ are in one (respectively two, six) partition(s), we say S' is S-1-*completable* (respectively S-2-*completable*, S-6-*completable*).

Thus, if a maximal cap S is chosen, the 198 maximal caps disjoint from S are not all affinely equivalent when S is fixed as a set. The next proposition summarizes how a linear transformation that fixes S as a set permutes the maximal caps disjoint from S and the partitions containing S. These results were verified by applying linear transformations to caps and partitions.



(a) 1-completable. (b) 2-completable.

Fig. 8. Partitions of $AG(4, 3)$ containing S (in large black dots) and (a) a 1-completable cap and (b) a 2-completable cap (in large gray dots).

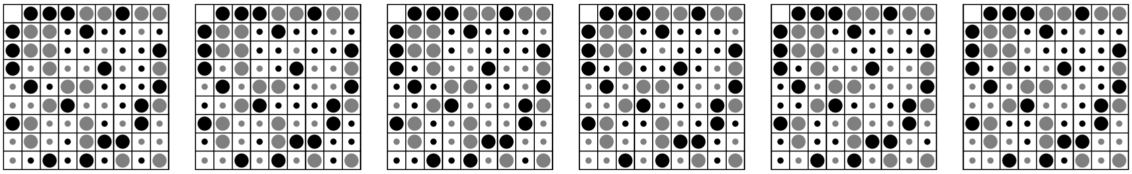


Fig. 9. Partitions of $AG(4, 3)$ containing S (in large black dots) and a 6-completable cap (in large gray dots).

Proposition 3.6. *Let S be a maximal cap with anchor $\vec{0}$, and let $\mathcal{T}$ be the group of linear transformations of $AG(4, 3)$ that fix S as a set; let $T \in \mathcal{T}$.*

(a) *If S_i is S-i-completable, then so is $T(S_i)$, for $i = 1, 2, 6$.*
(b) *Any partition of $AG(4, 3)$ containing S will have two S-*6-completable *caps and either an S-*1-completable *cap or an S-*2-completable *cap.*
(c) *The 216 partitions of $AG(4, 3)$ containing S are in two different equivalence classes E_1 and E_2 under the action of $\mathcal{T}$. E_1 contains 36 partitions that consist of $\{\{\vec{0}\}, S, S_1, S_{61}, S'_{61}\}$, where S_1 is S-*1-completable *and S_{61} and S'_{61} are both S-*6-completable. *E_2 contains 180 partitions that consist of $\{\{\vec{0}\}, S, S_2, S_{62}, S'_{62}\}$, where S_2 is S-*2-completable *and S_{62} and S'_{62} are both S-*6-completable.
(d) *$\mathcal{T}$ acts transitively on E_1 and acts transitively on E_2. If $\Pi = \{\{\vec{0}\}, S, A, B, C\}$ and $\Pi' = \{\{\vec{0}\}, S, A', B', C'\}$ and either $\Pi, \Pi' \in E_1$ with A and A' S-*1-completable, *or $\Pi, \Pi' \in E_2$, with A' and $A'S$-*2-completable, *then half the matrices in $\mathcal{T}$ that fix S and send A to A' (and thus sent Π to Π') send B to B' and C to C' and half send B to C' and C to B'.*
(e) *An S-*6-completable *cap appears in exactly one partition in E_1 and in five partitions in E_2.*

We have been considering partitions containing a particular maximal cap S with anchor $\vec{0}$. Because all maximal caps are affinely equivalent, this was sufficient (and much more convenient) for analyzing the structure of the partitions and the group action. We now broaden our perspective to consider all partitions with anchor $\vec{0}$, which will extend by translation to all partitions.

How many partitions are there with anchor $\vec{0}$? These partitions are not all linearly equivalent, so how many equivalence classes are there? There are 8424 caps we could have chosen as our fixed cap and 216 partitions containing that cap, and multiplying these numbers counts each partition 4 times, so there are 454,896 partitions with anchor $\vec{0}$. These partitions are acted on by the full general linear group, $GL(4, 3)$, which has order 24,261,120. Because 454,896 does not divide 24,261,120, the partitions must be in at least two equivalence classes. To understand these equivalence classes, we need to understand how the caps in the partitions behave with respect to each other.

Let $\{\{\vec{0}\}, A, B, C, D\}$ be a partition of $AG(4, 3)$ into 4 mutually disjoint maximal caps together with their anchor point. Clearly, if B is A-1-completable (respectively A-2-completable, A-6-completable), then A is B-1-completable (respectively B-2-completable, B-6-completable). This motivates the next definition.

Definition. Let $\{\{\vec{0}\}, A, B, C, D\}$ be a partition of $AG(4, 3)$ into 4 mutually disjoint maximal caps together with their anchor point. We say $\{A, B\}$ is a 1-*completable pair* (respectively a 2-*completable pair*) if the set $\{A, B\}$ appears in exactly one partition (respectively exactly two partitions).

Suppose that $\{\{\vec{0}\}, A, B, C, D\}$ is a partition where C and D are A-6-completable. The next lemma shows that C and D are themselves a 1-completable pair or a 2-completable pair. Further, a partition of $AG(4, 3)$ into 4 mutually disjoint maximal caps and their anchor point must consist of two pairs of caps, where either both pairs are 1-completable or both are 2-completable. (Note that this means that the other pair has both caps S-6-completable with respect to either cap S in the first pair.)

Lemma 3.7. *Let* $\{\{\vec{0}\}, A, B, C, D\}$ *be a partition of* $AG(4, 3)$ *into* 4 *mutually disjoint maximal caps together with their anchor point.*

(1) *If* B *is* A-1-completable *or* A-2-completable*, then* D *is* C-1-completable *or* C-2-completable.

(2) *Let* $\{\{\vec{0}\}, A, B, C, D\}$ *be a partition of* $AG(4, 3)$ *into* 4 *mutually disjoint maximal caps together with their anchor point. Then* $\{A, B\}$ *is a* 1-completable *pair if and only if* $\{C, D\}$ *is a* 1-completable *pair. Thus,* $\{A, B\}$ *is a* 2-completable *pair if and only if* $\{C, D\}$ *is a* 2-completable *pair.*

Proof. (1) Suppose that B is A-1-completable or A-2-completable. If we consider the partition $\{\{\vec{0}\}, B, A, C, D\}$ and think of B as the fixed maximal cap, we have a partition that can contain only one B-1- or B-2-completable cap, and since A is in only one or two partitions with B, we must have that C and D are both B-6-completable, so B is also C-6-completable (and D-6-completable). Since both A and B are 6-completable with respect to C and D, then considering the partition as fixing C, it must also be true that C and D are 1- or 2-completable with respect to each other.

(2) Given the partition $\{\{\vec{0}\}, A, B, C, D\}$, assume that $\{A, B\}$ is a 1-completable pair. Then C is 6-completable with respect to A. Let $\Pi_i = \{\{\vec{0}\}, A, B_i, C, D_i\}$, $2 \le i \le 6$ be the five additional partitions containing A and C. Then B is the unique 1-completable cap with respect to A among those partitions (by Proposition 3.6(e)), so WLOG, $\{A, B_2\}, \dots, \{A, B_6\}$ are 2-completable pairs.

By Proposition 3.6(d), there are linear transformations T_i fixing A and sending Π_2 to Π_i, $i = 3, \dots, 6$ that fix C as well. Shifting our point of view so that we are thinking of these partitions as fixing C, by Proposition 3.6(a) and (e), the existence of the transformations T_i imply that $D_2, \dots, D_6$ must be 2-completable with respect to C, so $\{C, D\}$ is a 1-completable pair. This means that the pairing of caps in any partition must have either two 1-completable pairs or two 2-completable pairs. □

We can now put these results together to give the equivalence classes of partitions of $AG(4, 3)$ with an arbitrary anchor point. The affine group acting on the elements of $AG(4, 3)$ is $\mathit{Aff}(4, \mathbb{F}_3) \cong GL(4, 3) \ltimes AG(4, 3)$. This action sends caps to caps, so it also sends partitions to partitions. Thus, we can extend the structures we have found for partitions with anchor $\vec{0}$ to all possible partitions of $AG(4, 3)$.

Theorem 3.8. *The partitions of* $AG(4, 3)$ *into* 4 *mutually disjoint maximal caps and the associated anchor point* $\vec{a}$ *are in two equivalence classes under the action of the affine group* $\mathit{Aff}(4, 3)$*. One equivalence class consists of partitions with two* 1-completable *pairs, and the other consists of partitions with two* 2-completable *pairs.*

Proof. From Lemma 3.7, all partitions consist of two 1-completable pairs or two 2-completable pairs. Extending Proposition 3.6(a), if we see that if $\{\{\vec{0}\}, A, B, C, D\}$ and $\{\{\vec{a}\}, A', B', C', D'\}$ are two partitions and $(T, \vec{a})$ is an element of $\mathit{Aff}(4, 3)$, taking A to A', etc., then B is 1-completable with respect to A if and only if B' is 1-completable with respect to A'. So, 1-completable pairs must go to 1-completable pairs, and 2-completable pairs must go to 2-completable pairs. Thus, one equivalence class under the action of $\mathit{Aff}(4, 3)$ is the set of partitions containing two 1-completable pairs; the other is the set of partitions containing two 2-completable pairs. □

4. Subgroups of the affine group acting on partitions

The full automorphism group of $AG(4, 3)$ is the group of affine transformations, the affine group $\mathit{Aff}(4, 3) = GL(4, 3) \ltimes \mathbb{Z}_3^4$. This group is of order 1,965,150,720. Consider $\vec{0}$ in $AG(4, 3)$ and its stabilizer $GL(4, 3)$, of order 24,261,120. Since $\mathit{Aff}(4, 3)$ is 2-transitive on points in $AG(4, 3)$, $GL(4, 3)$ is transitive on points. $GL(4, 3)$ is transitive on caps with anchor $\vec{0}$, but not 2-transitive on caps with anchor $\vec{0}$: while we can send any cap C with anchor $\vec{0}$ to another cap C' with anchor $\vec{0}$, we can only send C-1 completable (respectively C-2-completable, C-6-completable) caps to C'-1 completable (respectively C'-2-completable, C'-6-completable) caps, and that action extends to the action of $\mathit{Aff}(4, 3)$ on all caps. Thus, without loss of generality, we can understand the full group action by considering the stabilizer G of one particular maximal cap S as a set (so G also necessarily stabilizes $\vec{0}$), a subgroup of size 2880.

The results in this section were found using *Mathematica* [18] to compute with matrices and GAP [17] to analyze the structure of the groups of matrices. Recall, E_1 is the set of partitions containing a particular cap S (with anchor point $\vec{0}$), an S-1-completable cap and a second 1-completable pair (where both caps in that pair are S-6-completable); E_2 is the set of partitions containing S, an S-2-completable cap and another 2-completable pair (where both caps in that pair are S-6-completable).

G is transitive on the partitions in E_1 and transitive on the partitions in E_2. The subgroup G_1 of transformations of determinant 1 is transitive on the partitions in E_1 but not transitive on the partitions of E_2. If Π_2 is a partition in E_2, then $\{T(\Pi_2) \mid T \in G_1\}$, is half of the partitions of E_2, and each S-2-completable cap appears exactly once in that set. This means that each 2-completable cap also appears exactly once in $\{T'(\Pi_2) \mid T' \in G - G_1\}$.

Let Π_1 be a partition in E_1 and let S_1 be the S-1-completable cap in Π_1. There is a subgroup H of G of size 40 stabilizing the individual caps of Π_1 as sets. These transformations are all of determinant 1. H is nonabelian and has a unique subgroup isomorphic to $\mathbb{Z}_{20}$ and so is isomorphic to $\mathbb{Z}_{20} \rtimes \mathbb{Z}_2$. There are also 40 transformations that stabilize S and S_1 as sets and switch the two S-6-completable caps in the decomposition; these are all of determinant 2. Thus there is a group of order 80 stabilizing S and S_1 as sets.

Let Π_2 be a partition in E_2 and let S_2 be the S-2-completable cap in Π_2. There is a subgroup K of G of size 8 fixing the individual caps of Π_2 as sets; these transformations are all of determinant 1. K is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. There are also 8 transformations that stabilize S and S_2 as sets and switch the two S-6-completables in the decomposition; these are also all of determinant 1. This group of order 16 fixing Π_2 as a collection of caps is isomorphic to $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$. There is another set of 16 linear transformations that stabilize S and S_2, but which send the two S-6-completable caps in D_2 to the other 2-completable pair that appears in a partition with S and S_2. These transformations all have determinant 2. The group of order 32 stabilizing S and S_2 is isomorphic to $(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

Let S_6 be S-6-completable. Then exactly one of the partitions containing S and S_6 has an S-1-completable cap, from Proposition 3.6(e). The subgroup of G fixing S and S_6 is the same subgroup of order 40 that fixes S and the unique S-1-completable cap associated with S_6.

Finally, G has 144 elements of order 5, so there are 36 distinct subgroups isomorphic to $\mathbb{Z}_5$. Each of these subgroups fixes a unique element of E_1. Two elements of order 5 generate the subgroup containing all the elements of order 5, which is isomorphic to A_6.

How these subgroups permute the partitions and the 1- and 2-completable pairs could prove instructive in understanding the geometric structure that distinguishes the two classes of partitions.

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