COLOR-PERMUTING AUTOMORPHISMS OF CAYLEY GRAPHS

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Abstract. Given a group G with generators Δ, it is well-known that the set of color-preserving automorphisms of the Cayley color digraph $\Gamma = Cay_\Delta(G)$ is isomorphic to G. Many people have studied the question of when the full automorphism group of the Cayley digraph is isomorphic to G. This paper explores what happens when the full automorphism group of G is not isomorphic to G: how much larger it can be and what kinds of structures can be found. The group of automorphisms that permute the color classes is a semidirect product; we look more closely at that and other sub-structures.

1. Introduction

Automorphisms of Cayley graphs have been studied in many contexts. It is well known that if a Cayley graph has its edges colored corresponding to the generators, then the group of automorphisms which leave the colors unchanged is isomorphic to the original group. In [7], Fiol, Fiol and Yebra introduced the notion of color-permuting automorphisms, which are those which act as a permutation on the color classes. This paper extends their results and gives a preliminary examination of some of the group structures which can arise for the group of color-permuting automorphisms as well as for the full group of digraph automorphisms of the Cayley graph.

If G is a group with generators $\Delta = \{h_1, h_2, \ldots, h_n\}$, then the *Cayley graph* Γ of G for Δ is a directed graph associated with G and Δ where the edges have been assigned colors; it is sometimes referred to as the Cayley color graph. Its vertex set consists of the group elements of G. Each generator h_i is assigned a color c_i; for two group elements g_1 and g_2, there will be an edge colored c_i directed from g_1 to g_2 if $g_2 = g_1 h_i$. If h_i is an element of order 2 in G, then the two edges between g and gh_i are usually drawn as a single undirected edge, colored c_i. Certainly, the Cayley graph will depend on the generators chosen. Different sets of generators for the same group can give rise to quite different graphs.

A *digraph automorphism* of a digraph D is a permutation of the vertices of D that preserves directed adjacencies, so there is a directed edge from u to v in D if and only if there is a directed edge from $\phi(u)$ to $\phi(v)$. The set of digraph automorphisms of D forms a group under composition, denoted $Aut(D)$.

If G is a nontrivial finite group with generators Δ, then an element of $Aut(\Gamma)$ is color-preserving if the colors of the edges after the vertex permutation are the same as they were before. Clearly, the set of color-preserving automorphisms is a

Key words and phrases. Automorphisms of Cayley graphs, color-permuting.

This research was supported by NSF Research Experience for Undergraduates (REU) grants DMS-912325 and DMS-0243763. The fifth author also had support from DIMACS; the second and fourth authors had Lafayette College EXCEL program grants.

subgroup of $Aut(\Gamma)$; this subgroup is isomorphic to G (see, for example, Theorem 4-8 in [14]). However, there can be automorphisms of the Cayley graph that are not color-preserving: in some cases, the colors will be consistent, so that all edges of a given color will be some other color, while in other cases, the colors will have changed so there is no discernible pattern. This situation gives rise to natural questions: When will there be non-color-permuting atuomorphisms? How much larger than the original group can the group of color-preserving automorphisms be? How much larger can the full automorphism group of the Cayley graph be? How do these groups relate to the original group G?

These questions take the discussion of automorphisms of the Cayley graph in a different direction than most previous work in the area. We will now give a brief survey of some of the previous work on automorphism groups of Cayley graphs and its relation to the work in this paper. Much work has been done on finding graphs with a given automorphism group; a graph X is called a *graphical regular representation* of a group G if the automorphism group of X is isomorphic to G and if it acts transitively on the vertex set of X. In [9], C. Godsil characterized the only finite groups without graphical regular representations. For these groups, all Cayley graphs will have non-color-preserving automorphisms. Others have studied the question of when the group of automorphisms of a Cayley graph of a group is isomorphic to that group. For example, in 1982, L. Babai and C. Godsil conjectured that unless a group G belongs to a known class of exceptions, the automorphism groups of almost all Cayley graphs of G are isomorphic to G [1]. This conjecture remains open. If the conjecture is true, the collection of Cayley graphs whose automorphism group is larger than the original group is small. In this paper, we examine more closely this collection. What is the structure of those Cayley graphs? When is it possible to have such graphs? Answers to these questions might shed light on the conjecture.

Other authors have considered this situation as well. W. Imrich and M. E. Watkins, in a 1976 paper [10], considered the full automorphism group of the Cayley graph of a group G. They were concerned with generating sets that were closed under inverses, so the Cayley graph is undirected; they wanted to minimize the index of G in Aut(Cay(G)) for a given group where the index is always greater than 1 (that is, those without a graphical regular representation). The techniques we develop here can be used to examine these questions in the directed case as well as the undirected case.

Other authors have looked at the undirected case as well. For example, in a 1952 paper [8], R. Frucht constructed a one-regular graph of degree three; while doing so, he proved that if a generating set w of a group G is invariant under a group automorphism π of G, then π is an automorphism of the Cayley graph of G with respect to w (in that paper, all his generators were of order 2, so all the Cayley graphs were undirected). In this paper, we extend that result to Cayley graphs for generators with larger order. A 1991 paper by U. Baumann [2] examined the relationship between automorphisms of groups and automorphisms of their Cayley graphs, in cases where all generators have order 2. Baumann extended Frucht's idea to identify a subgroup of the full graph automorphism group containing all automorphisms induced by the group automorphisms that leave the generating set invariant. In that paper, Theorem 1 states that if an automorphism of the Cayley graph permutes the colors, it must be induced by a group automorphism of the generators (although he didn't state the result in that form). We show here that the same result holds in the directed case.

In a 1992 paper [7], Fiol, Fiol and Yebra specifically defined color-permuting automorphisms. Their definition was a generalization of the idea of a chromatic automorphism of a vertex-colored graph defined by Chvátal and Sichler [5]. The goal in [7] was to give necessary and sufficient conditions for the arc-colored line digraph of a Cayley digraph also to be a Cayley digraph. In that paper, they mention (without proof) that G is a normal subgroup of the group of color-preserving automorphisms of the Cayley graph, which we cite here as part of Theorem 2.5. In [6], Fang, Praeger and Wang look at the (undirected) Cayley graph for finite simple groups, giving some sufficient conditions for G to be a normal subgroup of the full automorphism group of the Cayley graph. We will take this idea further in this paper, looking at the case where the Cayley graph is directed.

In 2000, R. Jajcay [11] considered undirected Cayley graphs of groups generated by generating sets X that are closed under inverses. In this paper, he showed that the full automorphism group of a Cayley graph could be described using rotary extensions, a generalization of split extensions or semidirect products. A rotary extension, in contrast to the semidirect product of a group H by another group $K \leq Aut(H)$, is an extension of H by a subgroup of the group of all permutations on H stabilizing the identity. He then demonstrated that the full automorphism group of the Cayley graph of G is a rotary extension of G with a subgroup of the stabilizer of the identity vertex in $Aut(Cay(G))$. His goal in doing so was to address the problem of classifying all finite groups G that are isomorphic to the full automorphism group of some graph (which is necessarily a Cayley graph), a problem closely related to the problem of graphical regular representations previously mentioned.

In [12], Praeger defines normal edge-transitive Cayley graphs as those where a subgroup of automorphisms normalizes G and acts transitively on the edges of G. In that paper, she shows that the Cayley graph is normal edge-transitive as an undirected graph when the automorphisms of G that fix the generators S as a set is transitive on S. This happens precisely when the normalizer of G in the group of automorphisms of the Cayley graph is a semi-direct product. This group is the same as the color-permuting automorphisms we discuss in this paper. In another paper [15], Xu calls a Cayley digraph normal if G is normal in the full automorphism group of the Cayley graph. In the paper, he gives examples where the Cayley graph is normal and finds some results.

In this paper, we focus on directed versions and extensions of many of these questions. In Section 2, we look at the subgroups consisting of color-preserving automorphisms, color-permuting automorphisms and the subgroup of automorphisms that fix the identity vertex, showing that the latter is a subgroup of products of symmetric groups; we also show how that structure is obtained. Next, in Section 3, we show that $Aut(\Gamma)$ is isomorphic to a semidirect product of G with A, the subgroup of automorphisms that fix the identity vertex. We also determine the structure of A, based on the relations for the generators. We also look at other subgroups that help determine the structure of the various automorphism groups. Finally, in Section 4, we list several open problems.

2. Automorphisms Preserving Colors, Permuting Colors and Fixing the Identity

Let G be a non-trivial finite group with generators $\Delta = \{h_1, h_2, \ldots, h_n\}$. Let Γ denote the Cayley graph of G determined by Δ, and let $Aut(\Gamma)$ denote the group of digraph automorphisms of Γ.

The next definition follows the terminology of Fiol, Fiol and Yebra in [7].

Definition 2.1. Let G be a nontrivial finite group with generators $\Delta = \{h_1, h_2, \ldots, h_n\}$. Let $\{c_1, \ldots, c_n\}$ be a set of colors.

- An element ϕ of $Aut(\Gamma)$ is called *color-preserving* if the edge from g_i to g_j is colored c_k if and only if the edge from $\phi(g_i)$ to $\phi(g_j)$ is colored c_k. We denote by $Aut_G(\Gamma)$ the collection of color-preserving automorphisms.
- An element ϕ of $Aut(\Gamma)$ is called *color-permuting* if there exists a permutation τ of $\{1, 2, \ldots, n\}$ so that whenever the edge from g_i to g_j is colored c_k, the edge from $\phi(g_i)$ to $\phi(g_j)$ is colored $c_{\tau(k)}$. We denote by $Aut_P(\Gamma)$ the collection of color-permuting automorphisms.

2.1. **The Subgroups of Color-Preserving and Color-Permuting Automorphisms.** If we represent left multiplication of the vertices by the element g as $\lambda_g : h \mapsto gh$, then clearly λ_g induces a color-preserving automorphism of the Cayley graph. Theorem 2.2 states that all color-preserving automorphisms arise in this fashion. The result is well-known; see, for instance, [14] (Theorem 4-8) or [4] (Theorem 9.7). This justifies the notation $Aut_G(\Gamma)$ for the color-preserving automorphisms of G.

Theorem 2.2. *Let G be a non-trivial, finite group with generators Δ. Let Γ be the Cayley graph of G for Δ. Then $Aut_G(\Gamma) \cong G$.*

Example 2.3. Consider the alternating group A_4 generated by $\Delta = \{(123), (142)\}$. The Cayley graph Γ is pictured in Figure 1.

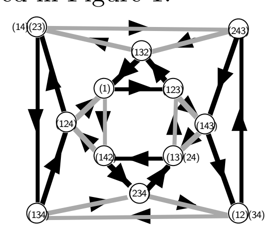
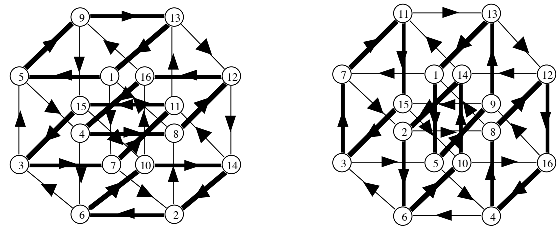


Figure 1. Cayley graph of A_4 in Example 2.3

$Aut_P(\Gamma)$ has 24 elements: there are 12 possible images for (1), and the image of (123) must be one of the two vertices adjacent to the image of (1). One choice will lead to a color-preserving automorphism and the other will lead to a color-permuting one.

Example 2.4. For an example of a Cayley graph which has non-color-preserving automorphisms, let H be the subgroup of S_8 generated by $\Delta = \{(1234)(56)(78), (12)(34)(5678)\}$. The Cayley graph Γ' of H is pictured in Figure 2a; it is a hypercube. In the figure, vertex 1 corresponds to the identity, vertex 5 corresponds to (1234)(56)(78), and vertex 7 corresponds to (12)(34)(5678).



(a) Γ' (b) $\phi(\Gamma')$

FIGURE 2. An automorphism which is not color-permuting.

For this graph, there are color-preserving and color-permuting automorphisms, but those only account for half of the automorphisms of this digraph. For example, the automorphism ϕ pictured in Figure 2b interchanges vertices 5 and 7, vertices 2 and 4, vertices 9 and 11, and vertices 14 and 16; it is not color-permuting.

We can ask the following questions:

(1) Is $Aut_G(\Gamma)) \lhd Aut_P(\Gamma)$?
(2) Is $Aut_G(\Gamma) \lhd Aut(\Gamma)$?
(3) Is $Aut_P(\Gamma) \lhd Aut(\Gamma)$?

By the following theorem, the answer to question (1) is yes. This result is mentioned in [7] without proof.

Theorem 2.5. *Let G be a non-trivial finite group with generators Δ and Cayley graph Γ for Δ. Then $G \cong Aut_G(\Gamma) \lhd Aut_P(\Gamma)$.*

Proof. Let σ represent any color-permuting automorphism in $Aut_P(\Gamma)$ and ϕ be a color-preserving automorphism in $Aut_G(\Gamma)$. Then $\sigma^{-1}\phi\sigma$ must be a color-preserving automorphism, since σ^{-1} will send an edge colored c_i to an edge colored c_j, ϕ will send an edge colored c_j to another edge colored c_j, and then σ will take an edge colored c_j to an edge colored c_i; the net effect is that all edges keep their original colors. □

For question (2), $Aut_G(\Gamma)$ may be normal in $Aut(\Gamma)$, but it needn't be. In [15], Xu called a Cayley digraph *normal* if $Aut_G(\Gamma) \lhd Aut(G)$. Subsequent papers and others have considered this question, which is far from settled. In [6], Fang, Praeger and Wang consider undirected Cayley graphs for certain finite simple groups, giving a sufficient condition for when G is a normal subgroup of the full automorphism group of the Cayley graph. As an example, in H of Example 2.4, the set of color-preserving automorphisms is not normal in the full automorphism group. If we designate by ψ the color-preserving automorphism which sends vertex 1 to vertex 5, then ψ is given by the vertex permutation $(1,5,9,13)$ $(2,6,10,14)(3,16,11,8)$ $(4,15,12,7)$. If we consider the same (non-color-permuting) automorphism ϕ from before, then the automorphism $\phi\psi\phi^{-1}$ is not color preserving.

For question (3), we again have that $Aut_P(\Gamma)$ may be normal in $Aut(\Gamma)$, but it needn't be. This question has not been studied previously, so the characterization of groups G and generating sets where this holds is still open. If the index $|Aut_P(\Gamma) : Aut(\Gamma)| = 2$ (as it is in Example 2.4), then certainly $Aut_P(\Gamma) \lhd Aut(\Gamma)$. However, if $G = D_4$ generated by $\Delta = \{(12), (34), (14)(23)\}$, then $Aut_P(\Gamma)$ is not normal in $Aut(\Gamma)$. The Cayley graph is shown in Figure 3. In this case,

$|Aut_P(\Gamma)| = 16$, generated by the color-preserving automorphisms and the color-permuting automorphism fixing the identity that interchanges the vertices corresponding to (12) and $(3,4)$. The full automorphism group of the Cayley graph is of order 48, the isometry group of the cube. It has no normal subgroup of order 16.

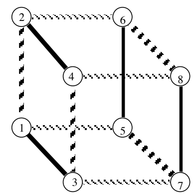


FIGURE 3. Cayley graph of D_4 generated by $\Delta = \{(12), (34), (14)(23)\}$

2.2. **The Color-Permuting Automorphisms that Fix the Identity Vertex.** If a color-permuting automorphism fixes the identity vertex, then that automorphism corresponds to an automorphism of G that permutes the generating set Δ. Thus, those generators must behave in the same way in the group relations. Bray, Curtis and others study symmetric presentations; some of that work has appeared in, for example, [3]. Our focus, however, is on how such a presentation corresponds to an automorphism of the Cayley graph.

Let A_P denote the subgroup of color-permuting automorphisms that fix the identity vertex, and let $Aut(G, \Delta)$ denote the group automorphisms of G that fix Δ as a set. We thus have that $A_P \cong Aut(G, \Delta)$.

For A_4 generated by $\Delta = \{(123), (142)\}$, the relations are $a^3 = b^3 = (ab)^3 = (1)$; note that $(ba)^3 = (1)$ as well. Thus, there will be color-permuting automorphisms. However, in H of Example 2.4, the presentation is $\langle a, b \mid a^4 = b^4 = (ab)^2 = (ab^{-1})^2 = 1\rangle$. One can verify that a and b are interchangeable in these relations, but it is also possible to change the colors of some edges only and have the new coloring represent the relations in different ways. For example, since $aaaa = abab = e$, the two edges corresponding to b in the 4-cycle $(abab)$ could have their color changed to that corresponding to a, and that 4-cycle would become the 4-cycle $(aaaa)$. At the same time, that would mean that the 4-cycle $(baba)$ becomes $(bbbb)$, $(bbbb)$ becomes $(abab)$, etc. In the example of ϕ given, that is exactly what happens, as one can verify from Figure 2.

For another example, consider the subgroup of S_6 generated by $\Delta = \{a_1, a_2, a_3\} = \{(123), (345), (156)\}$. Here, $a_i^3 = (1)$ for all i, $(a_i a_j)^5 = (1)$ if $i \neq j$ and $(abc)^5 = (1)$, yet $(bac)^4 = (1)$. Thus, we can only permute the generators in the Cayley graph with a $\mathbb{Z}_3$ action, not the full action of S_3. The Cayley graph for this group has 360 vertices and 1080 edges.

We summarize these ideas in the following theorem.

Theorem 2.6. *Let G be a non-trivial group with n generators Δ and let A_P denote the subgroup of automorphisms in $Aut_P(\Gamma)$ which fix the vertex corresponding to the group identity. Then $A_P \cong Aut(G, \Delta)$, a subgroup of S_n.*

We finish the section by putting these results together to describe the structure of the color-permuting automorphisms of the Cayley graph. Any color-permuting automorphism which sends the identity vertex to a vertex α can be written as a product of a color-permuting automorphism which fixes the identity followed by a color-preserving automorphism sending the identity vertex to α. That product can easily be shown to be a semidirect product.

Theorem 2.7. *Let G be a non-trivial finite group with n generators Δ. Then there is a subgroup A_P of $Aut_P(\Gamma)$ isomorphic to a subgroup of S_n such that $Aut_P(\Gamma) \cong G \rtimes A_P$.*

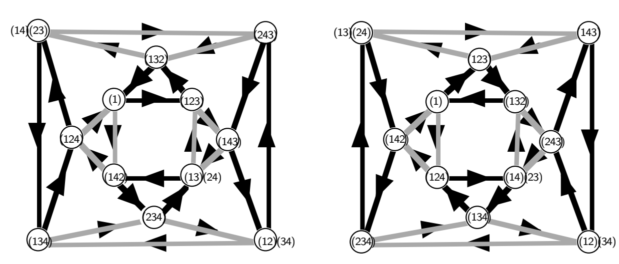
This theorem shows that $Aut_P(\Gamma)$ can always be expressed as a semidirect product of G with a subgroup of S_n that is determined by understanding how the generators can be interchanged in the group relations. As we will see in the next section, sometimes a stronger result occurs, where $Aut_P(\Gamma)$ will be a direct product with a group isomorphic to A.

This theorem is similar to one in [7]. In that paper, Fiol, Fiol and Yebra showed that the group of color-permuting automorphisms is isomorphic to a semidirect product of G with H^*, a subgroup of $Aut(G)$ which sends Δ to itself. However, they give no information about the structure of H^*.

Finally, note that all of the subgroups we have been discussing can be viewed as subgroups of the holomorph of G. Generally, the holomorph of a group G is the semidirect product $G \rtimes Aut(G)$, where the multiplication is given by $(g, \phi)(h, \psi) = (g\phi(h), \phi\psi)$. In our context, it is useful to view the holomorph as a permutation group acting on G; in this case, the holomorph is the normalizer of the subgroup of left regular representations of G in the full permutation group $Sym(G)$ of G. Because any automorphism of the Cayley graph corresponds to an element of the full permutation group of G, we can see how the groups we are interested in will be subgroups of $Sym(G)$. We will discuss this further in the next section.

3. Direct and Semidirect Products

For the Cayley graph A_4 generated by $\Delta = \{(123), (142)\}$ of Example 2.3, Theorem 2.7 shows that $Aut_P(\Gamma) \cong A_4 \rtimes S_2$. However, it is also true that $Aut_P(\Gamma) \cong A_4 \times S_2$. Why in this case can the semidirect product be decomposed (in a different way) into a direct product? In particular, if Γ' is the Cayley graph for A_4 generated by $\Delta_1 = \{(123), (124)\}$, then again, $Aut_P(\Gamma') \cong A_4 \rtimes S_2$, but in this case, the center is trivial, so it is not possible to express $Aut_P(\Gamma')$ as a direct product $A_4 \times S_2$. The two Cayley graphs are shown below in Figure 3.



(a) $Cay_\Delta(A_4)$ (b) $Cay_{\Delta_1}(A_4)$

Figure 3. Cayley graphs for two generating sets of A_4.

The difference here arises from the structure of $Aut(G, \Delta)$. In the first case, the element of $Aut(G, \Delta)$ that interchanges the two generators can be realized as conjugation by the element $(12)(34)$, which is an element of A_4, so it is an inner automorphism. In fact, the central element of $Aut(\Gamma)$ sends the vertex (1) to the vertex (12)(34). (One may visualize this as a central inversion of the directed-edge cuboctahedron that the Cayley graph represents.) On the other hand, the element of $Aut(G, \Delta')$ that interchanges the two generators can be realized as conjugation by the element $(3, 4)$, so it is not an inner automorphism. (The reader can verify that no central inversion of this directed-edge cuboctahedron preserves the directions of the arrows.)

This situation can be generalized. But first, we need several definitions.

3.1. **Inner Automorphisms and Centralizers.**

Definition 3.1. Let G be a group generated by Δ. Let A_I represent the elements of $Aut_P(\Gamma)$ that arise from inner automorphisms. That is, if $g \in G$ and $\gamma_g : h \mapsto ghg^{-1} = h^g$, $A_I = \{\gamma_g | g \in N(\Delta)\}$. Further, let $Aut_I(\Gamma) = Aut_G(\Gamma) \cdot A_I$.

Each γ_g is a color-permuting automorphism, since it fixes Δ as a set. However, any elements of $Z(G)$, the center of G, induce the identity automorphism. Thus, we have $A_I \cong N(\Delta)/Z(G)$. The following proposition shows how $Aut_I(\Gamma)$ fits between $Aut_G(\Gamma)$ and $Aut_P(\Gamma)$.

Proposition 3.2. *$Aut_G(\Gamma) \triangleleft Aut_I(\Gamma) \triangleleft Aut_P(\Gamma)$.*

Proof. The first assertion is immediate from Theorem 2.5. The proof of the second is the same as the proof that $Inn(G)$, the group of inner automorphisms of G, is a normal subgroup of $Aut(G)$. □

Example 3.3. Consider A_5 generated by $\Delta = \{(123), (124), (125)\}$. The generators can be permuted via conjugation by any permutation of $\{3, 4, 5\}$, however, only conjugation by 3-cycles will be inner automorphisms. Thus, $Aut_G(\Gamma) \cong A_5$, $Aut_I(\Gamma) \cong A_5 \times \mathbb{Z}_3$, and $Aut_P(\Gamma) \cong (A_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong A_5 \rtimes S_3$.

Right multiplication of vertices (in our context) does not usually produce an automorphism of the Cayley graph. However, in certain cases, it does.

Definition 3.4. Denote by ρ_g right multiplication by $g \in G$, that is, $\rho_g : h \mapsto hg$. Let $R = \{\rho_g | g \in N(\Delta)\}$, so $R \cong N(\Delta)$.

Proposition 3.5. (1) *$R = Cent_{Aut(\Gamma)}(Aut_G(\Gamma))$.*
(2) *$Aut_I(\Gamma) = R \cdot Aut_G(\Gamma)$.*

Proof.
(1) Follows immediately from the fact that the centralizer of G in $Sym(G)$ is the set of right multiplications; we are restricting our view to the subgroup $Aut(\Gamma)$.
(2) Follows from the definitions. □

$R \cap Aut_G(\Gamma) \cong Z(G)$; if $R \cap Aut_G(\Gamma) = \{e_G\}$, then A_I will be a direct product. When does that happen? $g \in R \cap Aut_G(\Gamma)$ precisely when $g \in Z(G)$, the center of G.

Corollary 3.6. *If $Aut_P(\Gamma) \cong Aut_G(\Gamma) \times H$ for some H, then $Aut_P(\Gamma) = Aut_I(\Gamma)$.*

Corollary 3.7. *If $Z(G) = e_G$, then $Aut_I(\Gamma) = Aut_G(\Gamma) \times R$.*

We have the following theorem that holds in the case where all automorphisms of G are inner automorphisms.

Theorem 3.8. *Consider the semidirect product $G \rtimes H$, where $Aut(G) = Inn(G)$ and multiplication is realized by $(x, a) \cdot (y, b) = (xy^a, ab)$. Suppose that in the map $\gamma : H \to Aut(G)$, $h \in H$ corresponds to conjugation by $g_h \in G$. Then (g_h, h^{-1}) is in the centralizer of G in $G \rtimes H$. Finally, if $H = \mathbb{Z}_2 \cong S_2$, then the element (g_h, h) is in the center of $G \rtimes S_2$.*

Proof. Let (a, e) be an element of $G \rtimes H$ corresponding to an element of G, and let (g_h, h) be as in the hypothesis of the theorem. We have that $a^h = g_h a g_h^{-1}$, $a^{h^{-1}} = g_h^{-1} a g_h$ and that $(g_h, h^{-1})^{-1} = ((g_h^{-1})^h, h)$. Then

$$\begin{aligned}(g_h, h^{-1})(a, e)(g_h, h^{-1})^{-1} &= (g_h a^{h^{-1}}, h^{-1})((g_h^{-1})^h, h)\\ &= (g_h(g_h^{-1} a g_h)((g_h^{-1})^h)^{h^{-1}}, hh^{-1})\\ &= (a g_h g_h^{-1}, e)\\ &= (a, e).\end{aligned}$$

The last statement of the theorem follows easily. □

4. Open Questions

(1) What groups can be realized as A, the color-permuting automorphisms that fix the identity (as in Theorem 2.6)?
(2) In general, when is the color-permuting automorphism group of a Cayley graph expressible as a direct product?
(3) Do our results hold for all alternating and symmetric groups? In particular, what happens with S_6, which has non-inner automorphisms?
(4) For which groups will the Cayley graph admit non-color-permuting automorphisms?
(5) What is the relationship of any group G with the full automorphism group of its Cayley graph? (Recall, G is not always normal in the full automorphism group.)

Acknowledgments. The authors wish to thank Derek Smith for useful suggestions and William Kocay for the program *Groups and Graphs* which generated the diagrams in this paper and which was very useful for discovering many of the relationships in this paper. We also wish to thank Gary Gordon and Lorenzo Traldi for useful feedback while the project was in development and on the final paper. Finally, Cheryl Praeger and Peter Cameron gave very helpful suggestions.

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