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Chordal graphs and the characteristic polynomial

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Abstract

A characteristic polynomial was recently defined for greedoids, generalizing the notion for matroids. When chordal graphs are viewed as antimatroids by shelling of simplicial vertices, the greedoid characteristic polynomial gives additional information about those graphs. In particular, the characteristic polynomial for a chordal graph is an alternating clique generating function and is expressible in terms of the clique decomposition of the graph. From it, one obtains an expression for the number of blocks in the graph in terms of clique sizes.

Keywords: Chordal graph; Characteristic polynomial; Antimatroid; Beta invariant; Greedoid

1. Introduction

Recently, a one-variable characteristic polynomial was defined for greedoids, extending the characteristic polynomial for matroids. The paper [6] contains a discussion of the history of the characteristic polynomial and some of its applications. Also, see [11] for an account of the characteristic polynomial of matroids, including discussion of Crapo's beta invariant (which we discuss in this paper as well). The characteristic polynomial traces its origins to the chromatic polynomial of a graph. In matroids, and in greedoids generally, there is no known coloring theory which corresponds to vertex colorings of graphs.

Greedoids were introduced by Korte and Lovász [8] to isolate the characteristics of matroids that allow the greedy algorithm to be an optimal strategy. Antimatroids are

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particular examples of greedoids; their structure is sufficiently interesting to have been rediscovered several times since they were first described by Dilworth. See [2] for an introduction to greedoids and antimatroids.

A greedoid Tutte polynomial was originally defined in [5]. Since then, that polynomial, the characteristic polynomial (which is an evaluation of the greedoid Tutte polynomial) and the beta invariant (an evaluation of the derivative of the characteristic polynomial) have been used to find new identities for various greedoids and antimatroids. For example, in [1], a new formula was obtained for the number of interior points of a finite set of points in the plane. In this paper, we continue that program, looking at the characteristic polynomial and the beta invariant for the simplicial shelling antimatroid on chordal graphs.

We begin Section 2 by defining antimatroids in terms of convex sets. In Section 3, we present several new results on the characteristic polynomial for general antimatroids which make use of the convex sets. In Section 4, we consider the characteristic polynomial of chordal graphs, showing that the characteristic polynomial is an alternating generating function for the number of cliques of each possible size. We also have a decomposition theorem which gives the characteristic polynomial in terms of the polynomials of the complete graphs which generate the chordal graph. We finish with a combinatorial identity giving the number of blocks of a chordal graph as an alternating sum involving the clique numbers.

2. Antimatroids

We now define antimatroids. There are several approaches to defining an antimatroid; since we will be concerned with convex sets in this paper, our definition will focus on those sets.

Definition. A *set system* over a finite ground set E is a pair $(E,\mathscr{C})$ where $\mathscr{C}$ is a collection of subsets of E ($\mathscr{C} \subseteq 2^E$). A set system $(E,\mathscr{C})$ is an *antimatroid* if it satisfies the following:

1. $\emptyset \in \mathscr{C}$, $E \in \mathscr{C}$,
2. for every $X \in \mathscr{C}$, $X \neq E$, there is an $e \notin X$ such that $X \cup \{e\} \in \mathscr{C}$,
3. for every $X \in \mathscr{C}$, $X \neq \emptyset$, there is an $e \in X$ such that $X - \{e\} \in \mathscr{C}$,
4. if $X_1, X_2 \in \mathscr{C}$, then $X_1 \cap X_2 \in \mathscr{C}$.

The sets in $\mathscr{C}$ are called the *convex sets* of the antimatroid. A convex set C is called *free* if every subset of C is also convex. We also define $|A|$ to be the size of the ground set of A.

Alternatively, one could define antimatroids by defining a closure operator.

Definition. A function τ defined on the subsets of a ground set E is a *closure operator* if it satisfies the following:

1. $A \subseteq \tau(A)$,

2. $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$,
3. $\tau(\tau(A)) = \tau(A)$.

The closure operator gives rise to a family of closed sets defined by $\mathscr{C} = \{X \subseteq E \mid X = \tau(X)\}$. In order for these closed sets to satisfy all the conditions for the convex sets of an antimatroid, two further conditions are needed:

4. $\tau(\emptyset) = \emptyset$,
5. If $y, z \notin \tau(X)$ and $z \in \tau(X \cup y)$, then $y \notin \tau(X \cup z)$.

Item 5 is the *antiexchange property* that contrasts with the exchange property for matroids and which gave rise to the name antimatroid.

A third way to define antimatroids is by identifying a special collection of sets called the *feasible* sets. Feasible sets are the complements of the convex sets. This is a common way of defining antimatroids; as we shall see, the name of the antimatroid defined on chordal graphs reflects this definition. A thorough discussion of the various ways of defining antimatroids can be found in [9].

When the characteristic polynomial is defined using feasible sets, a rank function is used, where the rank of a subset S of the ground set is defined to be the size of the largest feasible subset of S. Since the entire ground set of an antimatroid is feasible ($\emptyset \in \mathscr{C}$), the rank of the antimatroid is $|A|$, the size of the ground set.

3. The characteristic polynomial of an antimatroid

Let A be an antimatroid. The *characteristic polynomial* $p(A)$ was originally defined in [6]. The following formulation of the polynomial was Proposition 7 in that paper.

Proposition 3.1 (Free convex set expansion). *Let $A = (E, \mathscr{C})$ be an antimatroid and let $\mathscr{C}_{\mathrm{F}}$ be the collection of free convex sets. Then,*

$$p(A) = (-1)^{|E|} \sum_{K \in \mathscr{C}_{\mathrm{F}}} (-1)^{|K|} \lambda^{|K|}.$$

We now give some general results for antimatroids.

Theorem 3.1. *Let $A = (E, \mathscr{C})$ be an antimatroid, and let $|E| = n$. $p(A) = (-1)^{(n-k)} (\lambda - 1)^k$ if and only if there is a unique maximal free convex set and it is of size k.*

Proof. Every free convex set can be augmented to a maximal free convex set, so if there is a unique maximal free convex set X, then $\mathscr{C}_{\mathrm{F}}$ is the power set of X. Both directions of the theorem follow immediately from this fact and Proposition 3.1. □

The characteristic polynomial always has at least one factor of $\lambda - 1$ (Proposition 5 from [6]); it is a natural question to ask whether the number of factors of $\lambda - 1$ has any significance. We will show that, in the case of chordal graphs, the answer is yes. First, however, we have the following preliminary result.

Definition. In an antimatroid, a *max point* is an element that is in every maximal free convex set.

Theorem 3.2. *Let A be an antimatroid. Let $p(A)=(\lambda-1)^n q(\lambda)$, where $(\lambda-1)$ does not divide $q(\lambda)$. Then $n \geqslant$ the number of max points.*

Proof. We proceed using the free convex set expansion given in Proposition 3.1. Suppose v is a max point. Every free convex set can be extended to a maximal free convex set; thus, v can be added to any free convex set that does not contain it or deleted from any free convex set that does contain it, and the resulting set will be free convex as well. This means that there is a natural bijection between the free convex sets that contain v and those that do not.

Using the free convex set expansion and the remarks above, we have

$$\begin{aligned} p(A) &= (-1)^{|A|} \sum_{K \in \mathscr{C}_{\mathrm{F}}} (-1)^{|K|} \lambda^{|K|} \\ &= (-1)^{|A|+1}(\lambda-1) \sum_{v \notin K \in \mathscr{C}_{\mathrm{F}}} (-1)^{|K|} \lambda^{|K|} \\ &= (\lambda-1)(-1)^{|A|-1} p(A^*), \end{aligned}$$

where A^* is the antimatroid obtained by removing v from A. By induction, for each max point, we will obtain a factor of $\lambda - 1$, which gives the result desired. □

4. Chordal graphs

In this section, we examine the characteristic polynomial for chordal graphs. Chordal graphs are an important class of graphs, in part because they form a non-trivial class of perfect graphs. The characteristic polynomial for these graphs gives important insights into the structure of the graphs, as we will see in Theorem 4.2. We also use the beta invariant to obtain a purely graph-theoretic theorem about the number of blocks in the graph.

We begin with some basic definitions.

Definition. A graph is *chordal* if every cycle of length greater than 3 has a chord subdividing the cycle. A vertex of a graph is *simplicial* if its neighbors form a complete subgraph.

Any chordal graph has at least 2 simplicial vertices. See [3] or [7] for this result and a summary of other results on simplicial vertices and convexity in chordal graphs. Another property, which will turn out to be of importance here, is that every chordal graph has a clique decomposition. A summary of results in this area can be found in [10].

The next definition gives a standard antimatroid structure for chordal graphs. It is not particularly useful to us, because it depends on the feasible sets rather than the convex sets, so we will follow the definition by a characterization of the convex sets.

Definition. The *simplicial shelling antimatroid* $A(G)$ for a chordal graph G is defined by repeated elimination (shelling) of simplicial vertices and all edges incident with them. To be more precise, the feasible sets of $A(G)$ are sets of vertices which can be ordered so that the first vertex is simplicial in G, the second vertex is simplicial when the first vertex and its incident edges are eliminated, the third vertex is simplicial when the first two vertices and their incident edges are eliminated, etc.

Lemma 4.1. *If G is a connected chordal graph, then a subset C of vertices of G corresponds to a free convex set in $A(G)$ if and only if the vertices of C induce a clique in G. If G is a chordal graph which is not connected, then a subset C of vertices in G corresponds to a free convex set in $A(G)$ if and only if those vertices of C which lie in any connected component induce a clique in that component.*

Proof. The first statement is Lemma 5.1 in [4]. For the second statement, note that a free convex set is the complement of a feasible set. The feasible sets arise from the repeated shelling of simplicial vertices, and a vertex is simplicial in the entire graph if it is simplicial in the component. Thus, in any given connected component, the intersection of that component with a feasible set must be feasible in the connected component viewed as a chordal graph on its own. Thus, the same statement is true of the convex sets. □

Theorem 4.1. *Let G be a connected chordal graph with n vertices. Let K_i be the number of cliques of size i in G. Then*

$$p(A(G)) = (-1)^n \sum_{i=0}^{n} (-1)^i K_i \lambda^i.$$

Proof. This follows immediately from Proposition 3.1 and Lemma 4.1. □

We can put Lemma 4.1 and Theorem 4.1 together to tell us how to find the characteristic polynomial for a disconnected chordal graph G. This corollary allows us to be concerned in the remaining theorems only with connected chordal graphs without losing generality.

Corollary 4.1. *Let G be a chordal graph with n connected components, $G_1, G_2, \ldots, G_n$. Then $p(A(G)) = p(A(G_1))\,p(A(G_2))\cdots p(A(G_n))$.*

We begin our investigation of the characteristic polynomial of chordal graphs with the characteristic polynomial of a complete graph. This result is a corollary to Theorem 3.1, since a complete graph has a unique maximal free convex set.

Corollary 4.2. *Let K_n represent a complete graph on n vertices. Then*

$$p(A(K_n)) = (\lambda - 1)^n.$$

We next turn our attention to how chordal graphs arise by pasting complete graphs together, and how this pasting affects the characteristic polynomial. This result is particularly useful since every chordal graph is obtained by successive pasting. We say a graph G *arises from* G_1 *and* G_2 *by pasting along* S if $G = G_1 \cup G_2$ and $S = G_1 \cap G_2$. A graph is chordal if and only if it can be constructed recursively by pasting complete graphs along complete subgraphs. In this case, the graphs G_i are called the *simplicial summands* of G. Further, the set of simplicial summands obtained in a decomposition is independent of the order in which the decomposition is done. (See, for example [3, Proposition 5.5.1], and [10, Proposition 4.1].) This decomposition gives a recursive procedure for computing the characteristic polynomial which comes directly from the recursive pasting construction of the graph. We can then use that idea to express the characteristic polynomial simply, using the polynomials of the complete graphs that make up the chordal graph.

For ease of notation, when G_1 and G_2 are pasted along S, we will write $G_1 \cup_S G_2$. When there are multiple pastings, we will write the pastings in order from left to right, so we can dispense with parentheses.

Theorem 4.2. 1. *Let G_1 and G_2 be connected chordal graphs with n_1 and n_2 vertices, respectively. Let G be the chordal graph which arises from G_1 and G_2 by pasting along K_r, a complete graph on r vertices. Then G has $n = n_1 + n_2 - r$ vertices, and*

$$p(A(G)) = (-1)^n[(-1)^{n_1} p(A(G_1)) + (-1)^{n_2} p(A(G_2)) + (-1)^{r+1}(\lambda - 1)^r].$$

2. *Suppose the chordal graph G has been constructed recursively by pasting, so G can be expressed as $R_1 \cup_{S_1} R_2 \cup_{S_2} \cdots \cup_{S_{m-1}} R_m$. Suppose further that R_1 is a complete graph on r_1 vertices, $\ldots, R_m$ is a complete graph on r_m vertices, and S_1 is a complete graph on s_1 vertices, $\ldots, S_{m-1}$ is a complete graph on s_{m-1} vertices. If G has n vertices, then*

$$p(A(G)) = (-1)^n \left[\sum_{i=1}^{m} (-1)^{r_i}(\lambda - 1)^{r_i} + \sum_{j=1}^{m-1} (-1)^{s_j+1}(\lambda - 1)^{s_j} \right].$$

Proof. 1. Let G, G_1, G_2 and K_r be as in the statement of the theorem. The number of cliques of size i in G is equal to the number of cliques of size i in G_1 plus the number of cliques of size i in G_2 minus the number of cliques of size i in K_r. The result then follows from Theorem 4.1.

2. This follows by repeated applications of part 1 of the theorem, rewritten as $(-1)^n p(A(G)) = (-1)^{n_1} p(A(G_1)) + (-1)^{n_2} p(A(G_2)) + (-1)^{r+1}(\lambda - 1)^r$. As the pasting is done successively, the terms $(-1)^{n_k} p(A(G_k))$ are replaced either by previously computed terms or by $(-1)^{r_k}(\lambda - 1)^{r_k}$. □

We see, then, that we cannot reconstruct the graph from the polynomial, just as we cannot reconstruct the graph from knowing the sizes of the graphs that are pasted. In fact, using the language of Theorem 4.2, any two chordal graphs whose collections of graphs R_i and S_j are the same, no matter how or in what order they are pasted, will have the same characteristic polynomial.

Another consequence of this theorem is the following corollary, giving the significance of the largest power of $\lambda - 1$ which can be factored from the characteristic polynomial. This follows immediately from Theorem 4.2, Part 2. The case where $G = K_n$ is covered by Corollary 4.2, so we assume that the graph G has a non-trivial decomposition.

Corollary 4.3. *Let G be a connected chordal graph, $G \neq K_n$. Let m be the largest integer such that $p(A(G)) = (\lambda - 1)^m p_1(\lambda)$. If the decomposition of G is $G = R_1 \cup_{S_1} R_2 \cup_{S_2} \cdots \cup_{S_{n-1}} R_n$, where each R_i and S_j is a complete graph, then m is the size of the smallest S_j where pasting occurs.*

Examples. We will now compute the characteristic polynomial for several chordal graphs. In each case, we will give the decomposition of the graph, so that we can verify Theorem 4.2. The characteristic polynomial for each can be computed by counting the number of cliques and using Proposition 3.1 or by using Theorem 4.2. We will give only one form of the polynomial and leave it to the reader to verify that the other way of computing the polynomial gives the same result.

1. The most sparse chordal graphs are trees. If T is any tree on n vertices, then $p(A(T)) = (-1)^n(\lambda - 1)((n-1)\lambda - 1)$. Any tree can be decomposed as $K_2 \cup_{K_1} K_2 \cup_{K_1} \cdots \cup_{K_1} K_2$.

On the other extreme are chordal graphs which are a few edges short of complete graphs. We will now examine several examples of these kinds of chordal graphs.

2. If G is a complete graph minus an edge, $G = K_n \backslash \{e\}$, then $p(A(G)) = -(\lambda - 1)^{n-2}(2\lambda - 1)$. In this case, $G = K_{n-1} \cup_{K_{n-2}} K_{n-1}$.

3. In order to remove two edges from a complete graph and have the graph remain chordal, the two edges must share a vertex. In this case, $p(A(G)) = -(\lambda - 1)^{n-3}(\lambda^2 - 3\lambda + 1)$. In this case, we have $G = K_{n-1} \cup_{K_{n-3}} K_{n-2}$.

4. There are three ways that three edges can be removed from a complete graph and have the resulting graph remain chordal (see Fig. 1). Let $G_1(n,3)$ be a complete graph on n vertices with three edges removed, all of which are incident with the same vertex. Let $G_2(n,3)$ be a complete graph on n vertices with one triangle removed. Let $G_3(n,3)$ be a complete graph on n vertices with three edges removed which form a path.

Then, $p(A(G_1(n,3))) = -(\lambda - 1)^{n-4}(\lambda^3 - 3\lambda^2 + 4\lambda - 1)$, and $p(A(G_2(n,3))) = p(A(G_3(n,3))) = (\lambda - 1)^{n-3}(3\lambda - 1)$.

The graph G_1 arises from pasting as $K_{n-1} \cup_{K_{n-4}} K_{n-3}$. The two graphs G_2 and G_3 arise as $K_{n-2} \cup_{K_{n-3}} K_{n-2} \cup_{K_{n-3}} K_{n-2}$. In the case of G_2, the K_{n-3}'s are all the same subgraph induced by the vertices $4, 5, 6, \ldots, n$, while for G_3, the pasting is on two different subgraphs, and the order of pasting matters.

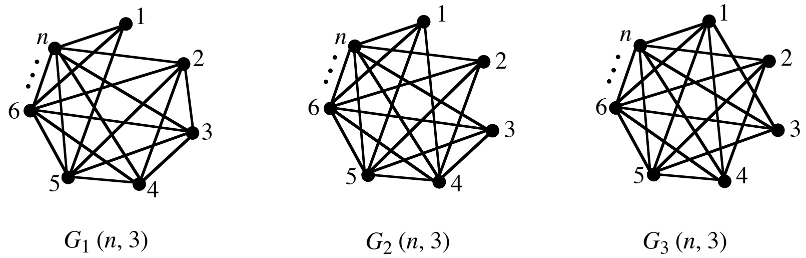

Fig. 1.

5. Two of these examples generalize. Let $G_1(n,r)$ be a complete graph on n vertices with r edges removed which are all incident to the same vertex, and let $G_2(n,r)$ be a complete graph on n vertices with one K_r removed. Then

$$p(A(G_1(n,r))) = -(\lambda-1)^{n-r-1}((\lambda-1)^r + (-1)^{r+1}\lambda),$$

$$p(A(G_2(n,r))) = (-1)^{r-1}(\lambda-1)^{n-r}(r\lambda-1).$$

$G_1(n,r)$ arises from pasting as $K_{n-1}\cup_{K_{n-r-1}} K_{n-r}$, while $G_2(n,r)$ is $K_{n-r+1}\cup_{K_{n-r}}\cdots\cup_{K_{n-r}} K_{n-r+1}$.

We conclude with a discussion of the significance of the β invariant for chordal graphs. The following definition is from [4].

Definition. Let A be an antimatroid with characteristic polynomial $p(A)=p(A;\lambda)$ (we rewrite the characteristic polynomial in this way to emphasize that it is a polynomial in λ). Then the β invariant, denoted $\beta(A)$, is defined as follows:

$$\beta(A) = (-1)^{|A|-1}p'(A;1).$$

Recall that a block in a graph is a maximal subgraph which contains no cut-vertex. Let $b(G)$ represent the number of blocks of a chordal graph G. In [4], Gordon showed that $\beta(A(G)) = 1-b(G)$. An immediate consequence of this fact is the following graph theoretic identity, giving the number of blocks in terms of the clique numbers.

Corollary 4.4. *Let G be a connected chordal graph with n vertices, and let k_i denote the number of cliques of size i in G. Then,*

$$b(G) = 1 + \sum_{i=1}^{n}(-1)^i i k_i.$$

Proof. The result follows from Theorems 4.1 and 5.1 in [4]. □

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