# On the Greedoid Polynomial for Rooted Graphs and Rooted Digraphs

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**ABSTRACT**

We examine some properties of the 2-variable greedoid polynomial $f(G; t, z)$ when $G$ is the branching greedoid associated to a rooted graph or a rooted directed graph. For rooted digraphs, we show a factoring property of $f(G; t, z)$ determines whether or not the rooted digraph has a directed cycle. © 1993 John Wiley & Sons, Inc.

## 1. INTRODUCTION

In [4], a two-variable polynomial $f(G; t, z)$ on greedoids was introduced. This paper will concentrate on the restriction of the polynomial to the greedoids associated with rooted graphs and rooted directed graphs. In Section 2, we examine the role of greedoid loops in rooted graphs and digraphs; in Theorem 1, we show that if a rooted graph or digraph has no greedoid loops, then it is completely determined by its greedoid structure. In Section 3, we consider factorization of the polynomial. We first show, in Proposition 3, that for a general greedoid, if the polynomial has a factor in $z$ alone, then that factor is a power of $z + 1$. We next examine the multiplicity of $z + 1$, first for general greedoids, then for rooted graphs, and finally for rooted digraphs. In particular, in Theorem 2, we show that the multiplicity of $z + 1$ in the polynomial for a rooted graph gives the number of greedoid loops in the rooted graph, while Theorem 3 shows that in a rooted digraph, the multiplicity is connected with greedoid loops and directed cycles. Section 4 looks at several examples of rooted digraphs and their polynomials to see what more can be said about multiplicities.

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We begin with some definitions. A *greedoid* $G = (E, r)$ consists of a ground set $E$, along with a rank function $r$: $2^E \rightarrow N \cup \{0\}$, such that for any $A, B \subseteq E$ and any $x, y \in E$,

R1. $r(A) \leq |A|$

R2. If $A \subseteq B$, then $r(A) \leq r(B)$

R3. If $r(A) = r(A \cup \{x\}) = r(A \cup \{y\})$, then $r(A) = r(A \cup \{x, y\})$.

A subset $F$ of E is called *feasible* if $r(F) = |F|$; in fact, a greedoid can also be defined by specifying the collection of feasible sets $F$ and then defining the rank function by $r(A) = \max\{|F|: F \subseteq A, F \in F\}$. For more information on greedoids, see [2]. Greedoids were introduced by Korte and Lovasz in a number of papers; see, for example, [5].

We now define the polynomial $f(G; t, z)$ (or $f(G)$ for short) by: $f(G) = \sum_{A \subseteq E} t^{r(G)-r(A)} z^{|A|-r(A)}$. This polynomial is a generalization of the (corank-nullity version of the) Tutte polynomial of a matroid; when the greedoid is actually a matroid, the new polynomial and the Tutte polynomial are identical. General information about the Tutte polynomials of matroids may be found in [3] or [7].

Some simple calculations show that if $G$ is a greedoid whose only element is feasible, then $f(G) = t + 1$, and if $G$ is a greedoid with $n$ elements, none of which is feasible (so $r(G) = 0$), then $f(G) = (z + 1)^n$.

If $G$ is a greedoid with an element $e$, then the *deletion* of $e$, $G - e$, is defined by specifying the feasible sets: $F_1 = \{F \subseteq E - e: F \in F\}$. If $\{e\}$ is feasible, then the *contraction* of $e$, $G/e$, is defined by specifying the feasible sets: $F_2 = \{F \subseteq E - e: (F \cup \{e\}) \in F\}$.

Proposition 1 and Proposition 2 are proven in [4].

**Proposition 1.** If $\{e\}$ is feasible, then $f(G) = f(G/e) + t^{r(G)-r(G-e)} f(G - e)$.

Define the *direct sum* of two greedoids $G_1$ and $G_2$ with disjoint ground sets $E_1$ and $E_2$ by specifying that the feasible sets of $G_1 \oplus G_2$ are precisely the disjoint unions of the feasible sets of $G_1$ and $G_2$.

**Proposition 2.** If $G$ is a greedoid with $G = G_1 \oplus G_2$, then $f(G) = f(G_1) \cdot f(G_2)$.

We now turn our attention to rooted graphs and rooted directed graphs. A *rooted graph G* is a graph with a distinguished vertex, denoted $*$. We call the set of edges of the graph $E(G)$ (or simply $E$) and the set of vertices $V(G)$ (or simply $V$). A rooted subgraph $T$ is *a rooted tree* if, for every vertex $v$ in $V$, there is a unique path in $T$ from $*$ to $v$. Then we can define a greedoid $\overline{G}$ with ground set $E$ and rank function $r$ given by $r(A) = \max\{|T|: T \subseteq A, T$ a rooted tree$\}$ for $A \subseteq E$. This is clearly equivalent to defining the feasible sets to be the edge-sets of rooted trees.

A *rooted directed graph D* is a digraph with a distinguished vertex, again denoted $*$. Again, we call the set of edges of the graph $E(D)$, or $E$, and the set of vertices $V(D)$, or $V$. A rooted subdigraph $A$ is called a *rooted arborescence* if, for every vertex $v$ in $V$, there is a unique directed path in $A$ from $*$ to $v$. Again, we define $\overline{D}$ to be the greedoid with ground set $E$ whose feasible sets are the edge-sets of rooted arborescences. In this paper, we will consider the greedoid polynomial on the associated greedoids of rooted graphs or rooted digraphs ($f(\overline{G})$ or $f(\overline{D})$). The greedoids associated with rooted graphs and rooted directed graphs are called branching greedoids and directed branching greedoids, respectively.

For both rooted graphs and rooted digraphs, we note that deletion of $e$ in the associated greedoid corresponds to removing the edge $e$ from the graph or digraph, i.e., $\overline{(G - e)} = \overline{G} - e$. If $e$ is an edge in $G$ emanating from $*$ (so that $\{e\}$ is feasible), contraction of $e$ in the associated greedoid corresponds to erasing the edge $e$ and identifying the two end points of $e$, i.e., $\overline{(G/e)} = \overline{G}/e$.

## 2. THE ROLE OF GREEDOID LOOPS IN ROOTED GRAPHS AND DIGRAPHS

Following the usual greedoid convention, we define a greedoid loop in a rooted graph or rooted digraph to be an edge that is in no feasible set of the associated greedoid. In a rooted graph, a greedoid loop can only be an edge from a vertex to itself (i.e., the usual graph-theoretic loop) or an edge in a part of the graph that is disconnected from $*$. In a rooted digraph, however, the situation is somewhat more complicated. A directed edge $e$ with initial vertex $v$ and terminal vertex $w$ is a greedoid loop if and only if $w$ lies on every directed path from $*$ to $v$. In Figure 1, for example, $e$ is a greedoid loop in each of the rooted digraphs $H_1$, $H_2$, $H_3$, and $H_4$.

No two of the four are isomorphic as rooted digraphs, but they all have isomorphic greedoid structures—hence their polynomials are the same. One can compute $f(H_i) = (z + 1)(t^2(z + 1) + (t + 1))$. Clearly, given a rooted graph or digraph, we can append greedoid loops in different ways to create nonisomorphic rooted graphs or digraphs; the underlying greedoid structure will only be able to distinguish how many greedoid loops have been

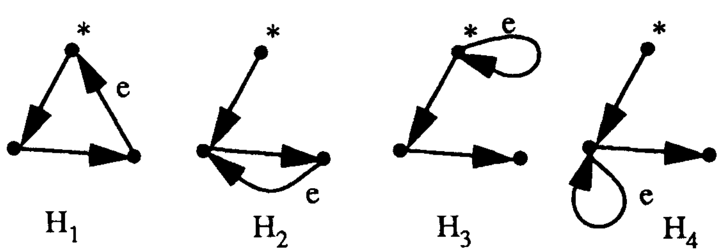


FIGURE 1.

added. The following theorem shows that if two rooted graphs or digraphs have the same greedoid structure, then the only way they could differ as graphs or digraphs is in the location of greedoid loops.

**Theorem 1.** A rooted graph (respectively, rooted digraph) with no greedoid loops and no isolated vertices can be uniquely reconstructed from the list of feasible sets of the associated greedoid. That is, if $G_1$ and $G_2$ are rooted graphs (digraphs) with no greedoid loops whose associated greedoids are isomorphic, then $G_1$ and $G_2$ are isomorphic as rooted graphs (digraphs).

***Proof.*** An outline of the proof follows. An alternate proof can be constructed from the fact that the poset of flats of the greedoid is the branching greedoid of the vertex search greedoid.

The result for rooted graphs follows by induction, either deleting one of a parallel edge emanating from $*$ (which can be recognized by the fact that $\{e\}$ and $\{f\}$ are each feasible in the associated greedoid but $\{e, f\}$ is not) or, if there are no such edges, by contracting an edge emanating from $*$ (an edge where $\{e\}$ is feasible).

For a rooted digraph $D$, one can first show that there must be an edge $e$ such that $D - e$ has no greedoid loops. This follows from the fact that if an edge $f$ becomes a greedoid loop when $e$ is deleted, then any edge that becomes a greedoid loop when $f$ is deleted also becomes a greedoid loop when $e$ is deleted.

To reconstruct $D$, delete an edge $e$ so that $D - e$ has no greedoid loops. Use induction to reconstruct $D - e$ from $\overline{D - e}$. The edge $e$ can be replaced in $D$ by finding its initial and terminal vertices; to determine the initial vertex, find a directed path containing $e$; to determine the terminal vertex, one can examine a basis $\overline{B}$ containing $e$ in $\overline{D}$ (which corresponds to a spanning arborescence $B$ in $D$) and then consider $B - e$ in $D - e$. Since, in a spanning arborescence, every vertex must have indegree 1, by considering three cases (either $B - e$ is a spanning arborescence for $D - e$, or it is connected but misses a vertex, or it is not connected in $D - e$), one can determine the terminal vertex of e. Thus, $D$ can be completely reconstructed.

This result can be compared with Whitney's 2-isomorphism theorem, which states that two graphs (each having no isolated vertices) give the same matroid structure if and only if the graphs are 2-isomorphic. See [6] for details.

## 3. FACTORIZATION OF THE POLYNOMIAL

We begin the discussion of the factorization of the polynomial with a result on general greedoids, and then look at the implications of this factorization for rooted graphs and rooted digraphs.

**Proposition 3.** Let $G$ be a greedoid. If $f(G;t,z) = f_1(t,z)g(z)$, then $g(z) = (z+1)^a$ for some $a \geq 0$.

***Proof.*** Let $G$ be a rank $r$, cardinality $n$ greedoid with $k$ feasible singletons. Note that $k \leq n$ and $r \leq n$. If $k = r = n$, then $f(G) = (t+1)^n$, so $g(z) = 1$. Next, if $k < n$, collect all terms in $f(G;t,z)$ corresponding to subsets $A$ of rank 0; this corresponds to all possible subsets of the set $S$ of $n-k$ nonfeasible singletons. For each such subset $A$, the term contributed to $f(G)$ is $t^r z^{|A|}$, so we have $\sum_{A \subseteq S} t^r z^{|A|} = t^r(z+1)^{n-k}$. Thus, we have $f(G;t,z) = t^r(z+1)^{n-k} + t^{r-1}q(z) + \cdots$. So, if $t^r(z+1)^{n-k} + t^{r-1}q(z) + \cdots = f_1(t,z)g(z)$, $g(z)$ must divide $(z+1)^{n-k}$, hence $g(z) = (z+1)^a$, as desired.

**Proposition 4.** If $G$ is a greedoid with $k$ loops, then $(z+1)^k$ divides $f(G)$.

***Proof.*** This follows immediately from the definition of $f(G)$ and Proposition 2.

**Lemma 1.** Let $G$ be a greedoid with $e \in G$ feasible. The multiplicity of $z+1$ in $f(G)$ will exactly equal the smaller multiplicity of $z+1$ in $f(G/e)$ or $f(G-e)$.

***Proof.*** Due to the recursion $f(G) = f(G/e) + t^{r(G)-r(G-e)} f(G-e)$, and the fact that $f(\text{loop}) = z+1$, we see by induction that the polynomial $f(G)$ can, in fact, be written as a polynomial with positive coefficients in $t$ and $(z+1)$. The lemma follows immediately.

The following theorem is exactly analogous to a result on the Tutte polynomial of a matroid.

**Theorem 2.** Let $G$ be a rooted graph with $f(\overline{G}) = (z+1)^a f_1(t,z)$, where $z+1$ does not divide $f_1(t,z)$. Then $a$ is the number of greedoid loops in $G$.

***Proof.*** The theorem is clearly true if $G$ has one edge, so assume it is true for all graphs with $n-1$ or fewer edges and that $G$ has $n \geq 2$ edges. For each greedoid loop in $G$, $f(\overline{G})$ has a factor of $z+1$, so we may assume $G$ has no greedoid loops and prove that $f(\overline{G})$ has no factors of $z+1$.

Let $e$ be an edge in $G$ that is incident with $*$. If the contraction $G/e$ has a greedoid loop, then there is an edge $f$ that is parallel with $e$, so there will only be a factor of $z+1$ in $G/e$ if $e$ is one of a set of parallel edges. However, in this case, $G-e$ has no greedoid loops, for the only way greedoid loops will appear in $G-e$ is for there to be edges incident to e that cannot be reached from $*$ by any other edges. Thus, either $G/e$ or $G-e$ has no greedoid loops. So, by induction, either $f(\overline{G/e})$ or $f(\overline{G-e})$ has no factors of $z+1$. Then, by Lemma 1, the proof is complete.

Recall that for a greedoid $G$, $f(G) = (z + 1)f(G - e)$, if $e$ is a loop. For a rooted directed graph, recall that a *directed cycle* is a sequence of edges $e_1, e_2, \ldots, e_n$, where the terminal vertex of $e_i$ is equal to the initial vertex of $e_{i+1}$, the terminal vertex of $e_n$ is equal to the initial vertex of $e_1$, and no vertex is visited twice along the way. We now show that the multiplicity of $z + 1$ in a rooted digraph is connected with directed cycles as well as greedoid loops.

**Theorem 3.** Let $D$ be a rooted directed graph with no greedoid loops. Then $D$ has a directed cycle if and only if $z + 1$ divides $f(\overline{D})$.

***Proof.*** First, we wish to show that if $D$ has no greedoid loops but does have a directed cycle, then $f(\overline{D})$ has a factor of $z + 1$. We again proceed by induction.

Any rooted digraph with three or fewer edges either has greedoid loops (and thus a factor of $z + 1$) or has no directed cycles, so the base case of the induction is vacuously satisfied.

Assume $D$ is a digraph with $n$ edges, $n \geq 4$; $D$ has no greedoid loops but does have a directed cycle. Suppose $e$ is a feasible edge with terminal vertex $v$. Clearly, a directed cycle in $D$ cannot contain $e$, as there are no greedoid loops in $D$. Thus, any directed cycle in $D$ will remain a directed cycle in the deletion $D - e$, possibly containing a greedoid loop. (See Figure 2: if $e$ is deleted, then $f$ becomes a greedoid loop.) Thus, by induction, $z + 1$ divides $f(\overline{D - e})$.

Next, if $D$ contains a directed cycle that does not pass through $v$, then it will remain a directed cycle in $D/e$. If the directed cycle does pass through $v$, then there must be an edge pointing into $v$, and hence there is a greedoid loop in $D/e$. By induction, $f(\overline{D/e})$ has factors of $z + 1$. Thus, by Lemma 1, $z + 1$ divides $f(\overline{D})$, as desired.

For the converse, we will show that if $D$ is a rooted directed graph with no greedoid loops and no directed cycles, then either $D - e$ or $D/e$ has no greedoid loops and no directed cycles. Using induction and Lemma 1, we will then have that $f(\overline{D})$ has no factors of $z + 1$. So, assume that $D$ is such a rooted directed graph.

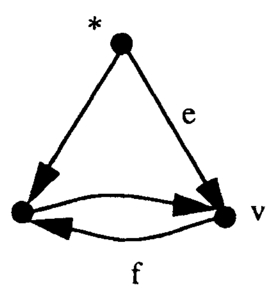


FIGURE 2.

If the number of edges in $D$ is 1, the result is clear, so assume that D has $n$ edges, where $n \geq 2$. Let $e$ in $D$ be an edge that is feasible in $\overline{D}$. (We know that such an edge must exist, for $D$ has no greedoid loops.) Thus, the initial vertex of $e$ is $*$; let $v$ be the terminal vertex of $e$.

Since $D$ has no directed cycles, neither $D/e$ nor $D - e$ can have a nontrivial directed cycle (that is, one not containing a greedoid loop). Thus, we need only show that either $D/e$ or $D - e$ has no greedoid loops.

Clearly, if $e$ is one of a set of parallel edges (emanating from $*$) or if there is an edge $f$ pointing into $v$, then contracting $e$ will create a greedoid loop. We will show that those are the only conditions under which contracting $e$ will create a greedoid loop. Suppose $e$ is not one of a set of parallel edges and $e'$ is an edge in $D$ that becomes a greedoid loop in $D/e$. Since $D$ has no greedoid loops, there is a shortest directed path from $*$ to the initial vertex of $e'$. If $e$ is in that path or if the path does not contain $v$, $e'$ could not be a greedoid loop in $D/e$, so the path does not contain $e$ but passes through $v$. But we could then use $e$ to give us a shorter path, contradicting minimality, unless $e'$ has $v$ as its terminal vertex. Hence, there is an edge pointing into $v$. Thus, $D/e$ will have a greedoid loop only if (1) $e$ is one of a set of parallel edges or (2) there is another edge pointing into $v$, the terminal vertex of $e$. We now show that under both of these conditions, $D - e$ has no greedoid loops.

In the case of condition (1), any greedoid loop of $D - e$ would be a greedoid loop in $D$, so $D - e$ has no greedoid loops.

In the case of condition (2), suppose $f$ is an edge whose terminal vertex is $v$. Let $e' \neq e$ be an edge in $D$. We will show that if $e'$ is a greedoid loop in $D - e$, then $D$ has a directed cycle, contrary to assumption. Since $D$ has no greedoid loops, there is a shortest directed path starting with $*$ containing $f$, say $e_1, e_2, \ldots, e_n, f$. (Note that $e$ cannot appear in that path.) There is also a shortest directed path starting with $*$ containing $e'$; however, this path must contain $e$, or else $e'$ would not be a greedoid loop in $D - e$. Thus, this path can be written $e, e_2', \ldots, e_m', e'$ (see Figure 3).

Now, consider the walk $e_1, e_2, \ldots, e_n, f, e_2', \ldots, e_m', e'$. This cannot be a feasible set (since $e'$ is a greedoid loop in $D - e$). This means that

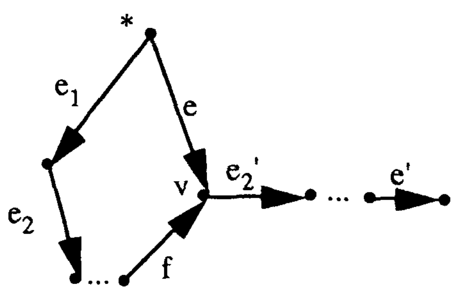


FIGURE 3

there is an $i \geq 2$ such that $e_1, e_2, \ldots, e_n, f, e_2', \ldots, e_i'$ is feasible, while $e_1, e_2, \ldots, e_n, f, e_2', \ldots, e_{i+1}'$ is not. Thus, $e_{i+1}'$ must have as its terminal vertex the initial vertex of one of the $e_j$. However, this now means that $e_j, \ldots, e_n, f, e_2', \ldots, e_{i+1}'$ is a directed cycle in $D$, which we cannot have. So, $e'$ cannot be a greedoid loop. Thus, $D - e$ has no greedoid loops.

Thus, in every case where $D/e$ has a greedoid loop, $D - e$ does not. By induction, either $f(\overline{D/e})$ or $f(\overline{D - e})$ has no factors of $z + 1$, so by Lemma 1, $f(\overline{D})$ has no factors of $z + 1$, and the proof is complete.

Theorems 6.7 and 6.10 of [1] together imply a result very similar to Theorem 3. In our notation, this result can be written as follows: $f(\overline{D}; 0, -1) = 0$ if and only if $D$ has no directed cycle. It is not hard to show directly that $f(\overline{D}; 0, -1) = 0$ if and only if $f(\overline{D}; t, -1) \equiv 0$, and thus derive Theorem 3. We point out, however, that the proofs of Theorems 6.7 and 6.10 in [1] make extensive use of combinatorial topology, and in particular depend on shellability properties of the dual complex of the branching greedoid. The proof we have given, on the other hand, is entirely graph theoretic.

## 4. COUNTEREXAMPLES

Let $D$ be a rooted directed graph, with no greedoid loops. If $D$ has a directed cycle, then Theorem 3 shows that there will be at least one factor of $z + 1$ in $f(D)$. It thus seems reasonable to expect that the multiplicity of $z + 1$ is connected in some way with the number of directed cycles. The following collection of counterexamples shows, among other things, that a straightforward answer (the number of directed cycles or the maximum number of disjoint directed cycles) will not suffice. Also, by considering examples of nonisomorphic rooted digraphs with the same polynomial, we will show that certain other digraph invariants are not determined by the polynomial $f(D)$.

All three digraphs in Figure 4 have rank 3 and six edges. $D_1$ has a directed cycle of length 2 and one of length 3, which have one edge in common. $D_2$ has two directed cycles of length 3, which have two edges in common. $D_3$ has a greedoid loop and no directed cycles. Now,

$$f(D_1) = f(D_2) = f(D_3) = (z + 1)[(z + 1)^3 t^3 + (z + 1)^2(t^2 + t + 1) + (z + 1)(t^2 + 2t + 2) + (t + 1)].$$

We see that the one factor of $z + 1$ cannot be counting the number of nondisjoint directed cycles, and that the polynomial cannot distinguish whether there is a directed cycle or a greedoid loop, nor can it determine the length of the longest or shortest directed cycle. In addition, in the first two cases, reversing the direction of one of the edges does not change the polynomial. We can also see that $D_1$ and $D_2$ disprove the conjecture in [4],

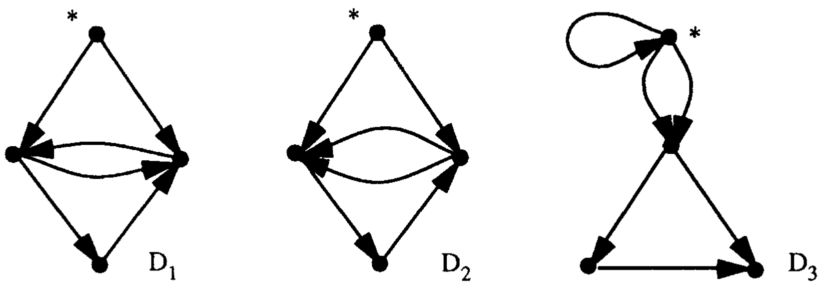


FIGURE 4

as every edge is in some feasible set and the digraphs are not isomorphic, yet they have the same polynomial.

The digraphs in Figure 5 both have rank 3 and seven edges. $D_4$ has four cycles of length 3, all of which have one edge in common; $D_5$ has one directed cycle of length 2 and a greedoid loop. Here,

$$f(D_4) = f(D_5) = (z + 1)^2[(z + 1)^3t^3 + (z + 1)^2(t^2 + t + 1) + (z + 1)(t^2 + t + 2) + 2(t + 1)].$$

There are several differences between these two, for example: for the first, the maximum number of disjoint directed cycles is one, yet $z + 1$ has multiplicity two, whereas the second has a greedoid loop, and there is only one directed cycle.

Figure 6 provides a nice contrast to Figure 5. In $D_6$, there are four directed cycles of length 2, two of which have one edge in common. There are also two directed cycles of length 4, which have three edges in common. When counting numbers of disjoint directed cycles, one can either count three (all 2-cycles), or one (a 4-cycle). In this case,

$$f(D_6) = (z + 1)^4[(z + 1)^4t^4 + (z + 1)^3t^3 + (z + 1)^2(t^3 + 2t^2 + 1) + (z + 1)(2t^2 + 2t + 4) + 4(t + 1)].$$

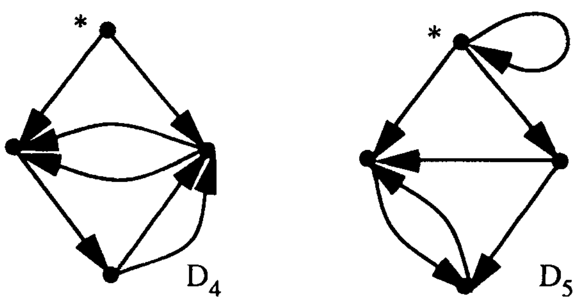


FIGURE 5

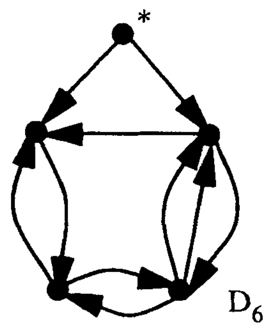


FIGURE 6

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