On the Structure of Mackey Functors and Tambara Functors

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Abstract

The stable homotopy groups of a G-spectrum are Mackey functors. Moreover, in 2004, Morten Brun showed that the zeroeth stable homotopy group of a commutative G-ring spectrum is a Mackey functor with added structure. More specifically, it is a Tambara functor. Thus, for G the cyclic group of prime power order, we endow the category of G-Mackey functors with a equivariant symmetric monoidal structure such that G-Tambara functors are the equivariant commutative monoids. This equivariant structure relies on the construction of symmetric monoidal norm functors from the category of H-Mackey functors to the category of G-Mackey functors for all subgroups H of G, and we devote most of Chapter 2 to defining these functors. We focus on the elegant and concrete nature of this new equivariant structure and provide numerous examples. We end by discussing some results on Tambara functors that follow directly from the computability of these norm functors.

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Contents

Chapter 1

Introduction

In algebraic topology we would often like generalized cohomology theories to remember the symmetries of a given topological space X. Thus, we equip X with an action of some finite group G and extend the definition of a generalized cohomology theory to an $RO(G)$-graded cohomology theory so that it reflects the group action. Just as a generalized cohomology theory can be represented by a spectrum, an $RO(G)$-graded cohomology theory can be represented by a G-spectrum, which is a spectrum appropriately endowed with an action of G. Therefore, many topologists have passed from studying topological spaces to studying G-spectra and hence developed the field of equivariant stable homotopy theory. For good references on this subject refer to [8], [10], and [3].

A large part of equivariant stable homotopy theory involves the study of the homotopy groups of G-spectra. However, the equivariant stable analogues of homotopy groups are more than just groups. Letting $\underline{S}^k$ be the k^{th} sphere spectrum and $[\underline{S}^k, X]$ denote the set of homotopy classes of maps $S^k \to X$, if X is a spectrum (with no

group action) then for all k in $\mathbb{Z}$ we define the k^{th} homotopy group of X by

$$\pi_k(X) = [\underline{S}^k, X].$$

Thus, we might guess that if we endow X with a G-action then we should define the k^{th} stable homotopy group of X by

$$\pi_k(X) = [\underline{S}^k, X]^G$$

where now $[\underline{S}^k, X]^G$ is the set of homotopy classes of G-equivariant maps $\underline{S}^k \to X$. However, this definition does not suffice because we require that the stable homotopy groups of a G-spectrum not only remember G-action information, but also record information about the action of all subgroups of G. Therefore, as described in Chapters 6 of [14] and 12 of [10], if X is a G-spectrum then to define the k^{th} stable homotopy group of X we must first consider

$$\pi_k^H(X) = [\underline{S}^k \wedge (G/H)_+, X]^G$$

for every subgroup H of G. Then as H varies these abelian groups fit together to form a *G-Mackey functor*, denoted $\underline{\pi}_k(X)$, and we define the k^{th} stable homotopy group of X to be this Mackey functor.

Moreover, a commutative G-ring spectrum is a G-spectrum with extra structure in the form of a commutative multiplication, and the zeroeth stable homotopy group of a commutative G-ring spectrum has extra structure as well. In fact, it has even more structure than one might initially expect. The collection of G-Mackey functors forms a

category, and this category has a symmetric monoidal product called the box product and denoted $\square$. A unital commutative monoid under the box product is called a *G-Green functor*, and so we might surmise that if X is a commutative G-ring spectrum then $\underline{\pi}_0(X)$ is a commutative G-Green functor. While this statement is not altogether false, it is incomplete since it does not describe the entirety of the structure of $\underline{\pi}_0(X)$. In fact, in 2004 Morten Brun proved that the zeroeth stable homotopy group of a commutative G-ring spectrum is a *G-Tambara functor* [1]. Thus, Brun proved that $\underline{\pi}_0(X)$ is a commutative G-Green functor with the extra structure of multiplicative transfer maps called norm maps.

It would therefore be beneficial to develop an equivariant symmetric monoidal structure on the category of Mackey functors that is reminiscent of the box product but such that Tambara functors become the equivariant commutative ring objects. Such a structure is called a *G-symmetric monoidal structure*, and the commutative ring objects are called *G-commutative monoids* [4]. We define this structure in terms of tensoring over finite G-sets. In particular, if we let $\mathscr{S}et_G^{Fin}$ be the category of finite G-sets and $\mathcal{M}ack_G$ be the category of G-Mackey functors then, in essence, a G-symmetric monoidal structure is a map

$$(-) \otimes (-) : \mathscr{S}et_G^{Fin} \times \mathcal{M}ack_G \to \mathcal{M}ack_G.$$

Then a given Mackey functor $\underline{M}$ is a G-commutative monoid if the map

$$(-) \otimes \underline{M} : \mathscr{S}et_G^{Fin} \to \mathcal{M}ack_G$$

extends to a functor. More specifically, the Mackey functor $\underline{M}$ is a G-commutative monoid if a map $X \to Y$ of finite G-sets induces a map $X \otimes \underline{M} \to Y \otimes \underline{M}$ of Mackey functors. We refer to Section 2.1 of this thesis for complete definitions of a G-symmetric monoidal structure and a G-commutative monoid.

Let G be C_{p^n}, the cyclic group of prime power order. The goal of this thesis is to endow the category of G-Mackey functors with a G-symmetric monoidal structure such that a given Mackey functor is a G-commutative monoid if and only if it has the structure of a Tambara functor. The G-symmetric monoidal structure that we create relies heavily on the construction of symmetric monoidal norm functors that build a G-Mackey functor from an H-Mackey functor for all subgroups H in G. Thus, most of this thesis consists of proving the following theorems.

Theorem. *There exist strong symmetric monoidal functors*

$$N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$$

for all subgroups H of G.

Theorem. *For all subgroups $H < K < G$ the functor N_H^G is isomorphic to the composition of functors $N_K^G N_H^K$.*

We then define the map $(-) \otimes (-)$ by

$$\begin{aligned} G/H \otimes \underline{M} &= N_H^G i_H^* \underline{M} \text{ for all subgroups } H \text{ of } G \text{ and} \\ (X \amalg Y) \otimes \underline{M} &= (X \otimes \underline{M}) \,\Box\, (Y \otimes \underline{M}) \text{ for all finite } G\text{-sets } X \text{ and } Y \end{aligned}$$

where the functor $i_H^* : \mathcal{M}ack_G \to \mathcal{M}ack_H$ is the restriction functor.

The category of G-Mackey functors already supports a G-symmetric monoidal structure [4]. However, this structure is difficult to unravel. Indeed, its definition requires the passage to G-spectra! Moreover, we have been unable to identify all Tambara functors as the G-commutative monoids under this structure. Therefore, there are two advantages of the new construction presented in this thesis. First, it is concrete and computable. In particular, we will define the norm functors (and thus a new G-symmetric monoidal structure) without passing to G-spectra. Instead, we will explicitly describe the functors using the algebraic and categorical properties of Mackey functors.

Finally, in Section 2.3.1 we prove the result given below.

Theorem. *Under this new G-symmetric monoidal structure a Mackey functor is a G-commutative monoid if and only if it has the extra structure of a Tambara functor.*

In the remaining sections of Chapter 1 we provide definitions and examples of G-Mackey functors, G-Green functors and G-Tambara functors for any finite abelian group G. We also describe the box product for C_{p^n}-Mackey functors and Tambara functors. We build the new C_{p^n}-symmetric monoidal structure on the category of C_{p^n}-Mackey functors in Chapter 2, and we devote Section 2.2 to manufacturing the norm functors. We end by discussing two interesting consequences of this equivariant symmetric monoidal structure. First, the norm functors will provide fun constructions of Tambara functors, and we also describe how to use the norm functors to endow a

commutative Green functor with more than one Tambara functor structure.

1.1 Mackey Functors

We give the definition of a Mackey functor that is due to Dress [2]. Let G be a finite abelian group, and let $\mathscr{S}et_G^{Fin}$ be the category of finite G-sets.

Definition 1.1.1. A *G-Mackey functor* $\underline{M}$ (or just *Mackey functor* when the group is clear) consists of a pair of functors (M_*, M^*) from $\mathscr{S}et_G^{Fin}$ to the category of abelian groups such that

- $M_*(X) = M^*(X)$ for all X in $\mathscr{S}et_G^{Fin}$. We denote this common value by $\underline{M}(X)$.
- $\underline{M}$ takes a disjoint union of finite G-sets to a direct sum of abelian groups.
- M_* is covariant and M^* is contravariant.
- Together, M_* and M^* take a pullback diagram in $\mathscr{S}et_G^{Fin}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ W \arrow[r, "k"'] & Z \end{tikzcd}$$

  to the following commutative diagram of abelian groups.

$$\begin{tikzcd} \underline{M}(X) \arrow[r, "M_*(f)"] & \underline{M}(Y) \\ \underline{M}(W) \arrow[u, "M^*(g)"] \arrow[r, "M_*(k)"'] & \underline{M}(Z) \arrow[u, "M^*(h)"'] \end{tikzcd}$$

Given a map $f : X \to Y$ of finite G-sets we call $M^*(f)$ a *restriction* map and $M_*(f)$ a *transfer* map.

We can develop a more concrete definition of a Mackey functor because any finite G-set can be written as the disjoint union of orbits G/H. Thus, we can completely understand a Mackey functor once we determine $\underline{M}(G/H)$ for all orbits and the restriction and transfer maps induced from maps between orbits. Since G is abelian a map $f : G/K \to G/H$ in $\mathscr{S}et_G^{Fin}$ only exists if K is a subgroup of H, and if K is a proper subgroup of H then we define any map $f : G/K \to G/H$ via the composition $G/K \xrightarrow{\phi} G/K \xrightarrow{\pi} G/H$ where π is the canonical quotient map and ϕ is an automorphism of G/K. Letting $W_G(K)$ be the Weyl group $N_G(K)/K$, distinct automorphisms of G/K are given by multiplication by γ for some γ in $W_G(K)$. It follows that for all subgroups K and H of G, all maps $G/K \to G/H$ are Weyl conjugate to the map π, and we only need to determine the transfer and restriction maps induced from π. We denote $M_*(\pi)$ by tr_K^H and $M^*(\pi)$ by res_K^H. Further, if $K' < K < H$ then the composition of quotient maps $G/K' \xrightarrow{\pi} G/K \xrightarrow{\pi} G/H$ must equal the quotient map $G/K' \xrightarrow{\pi} G/H$, and thus $tr_{K'}^H = tr_K^H tr_{K'}^K$ and $res_{K'}^H = res_{K'}^K res_K^H$.

Moreover, for all subgroups H of G the automorphisms of G/H induce an action of $W_G(H)$ on $\underline{M}(G/H)$, and since the quotient map $\pi : G/K \to G/H$ is a G-map the maps $tr_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ and $res_K^H : \underline{M}(G/H) \to \underline{M}(G/K)$ must be Weyl equivariant. Similarly, let $\gamma\cdot$ denote the automorphism of G/H given by the element

γ in $W_G(H)$. The commutativity of the diagram

$$\begin{tikzcd} G/H \arrow[r, "\pi"] \arrow[d, "\gamma\cdot"'] & G/G \arrow[d, "id"] \\ G/H \arrow[r, "\pi"] & G/G \end{tikzcd}$$

requires that every transfer map $tr_H^G : \underline{M}(G/H) \to \underline{M}(G/G)$ factor through the $W_G(H)$-action, and the codomain of every restriction map res_H^G consist of the $W_G(H)$-fixed points of $\underline{M}(G/H)$. More specifically,

$$tr_H^G(\gamma \cdot x) = tr_H^G(x) \text{ and } res_H^G(x) = \gamma \cdot res_H^G(x)$$

for all γ in $W_G(H)$.

Finally, the pullback requirement of Definition 1.1.1 forces the Weyl action to maintain additional structure. Every pullback diagram of orbits is of the form

$$\begin{tikzcd} G/H \times G/H \arrow[r, "p"] \arrow[d, "p"'] & G/H \arrow[d] \\ G/H \arrow[r] & G/G \end{tikzcd}$$

but we can rewrite $G/H \times G/H$ as the disjoint union

$$\coprod_{\gamma \in W_G(H)} (G/H)_\gamma$$

of $|W_G(H)|$-many copies of G/H. Under this identification one projection map p simply becomes the fold map ∇. However, the other becomes the fold map twisted by the $W_G(H)$-action, which we denote by ∇_γ, and thus, the pullback diagram becomes

$$\begin{tikzcd} \coprod_{\gamma \in W_G(H)} (G/H)_\gamma \arrow[r, "\nabla"] \arrow[d, "\nabla_\gamma"'] & G/H \arrow[d] \\ G/H \arrow[r] & G/G \end{tikzcd}$$

This diagram results in the following commutative diagram of abelian groups.

$$\begin{tikzcd} \bigoplus_{\gamma \in W_G(H)} \underline{M}(G/H) \arrow[r, "\nabla"] & \underline{M}(G/H) \\ \underline{M}(G/H) \arrow[u, "\Delta_\gamma"] \arrow[r, "tr_H^G"'] & \underline{M}(G/H) \arrow[u, "res_H^G"'] \end{tikzcd}$$

The map ∇ remains the fold map, and the map Δ_γ is the $W_G(H)$-twisted diagonal map given by $m \mapsto \bigoplus_{\gamma \in W_G(H)} \gamma \cdot m$. It follows that

$$res_H^G tr_H^G(x) = \sum_{\gamma \in W_G(H)} \gamma \cdot x$$

for all x in $\underline{M}(G/H)$ and subgroups H of G.

We can now give a more constructive yet equivalent definition of a Mackey functor.

Definition 1.1.2. A *Mackey functor* $\underline{M}$ consists of a collection of abelian groups $\underline{M}(G/H)$ along with transfer maps $tr_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ and restriction maps $res_K^H : \underline{M}(G/H) \to \underline{M}(G/K)$ for all subgroups $K < H \leq G$ such that the following relations hold.

1. If $K' < K < H$ then $tr_{K'}^H = tr_K^H tr_{K'}^K$ and $res_{K'}^H = res_{K'}^K res_K^H$.

2. If $K < H \leq G$ then there is an action of $W_H(K)$ on $\underline{M}(G/K)$ such that

   - $tr_K^H(\gamma \cdot x) = tr_K^H(x)$ for all x in $\underline{M}(G/K)$ and γ in $W_H(K)$,
   - $\gamma \cdot res_K^H(x) = res_K^H(x)$ for all x in $\underline{M}(G/H)$ and γ in $W_H(K)$, and
   - for all subgroups K and K' in H

$$res_{K'}^H tr_K^H(X) = \sum_{\gamma \in W_H(K')} \gamma \cdot tr_{K' \cap K}^{K'}(x)$$

for all x in $\underline{M}(G/(K' \cap K))$. In particular, if $K = K'$ then

$$res^H_K tr^H_K(x) = \sum_{\gamma \in W_H(K)} \gamma \cdot x$$

for all x in $\underline{M}(G/K)$.

Additionally, we will describe a Mackey functor via a *Mackey functor diagram*. For example, a C_8-Mackey functor is pictured in Figure 1.1 where C_8 is the cyclic group of order 8. We do not include the Weyl action in these diagrams.

$$\begin{tikzcd}
\underline{M}(C_8/C_8) \arrow[d, bend right, "res^{C_8}_{C_4}"'] \\
\underline{M}(C_8/C_4) \arrow[u, bend right, "tr^{C_8}_{C_4}"'] \arrow[d, bend right, "res^{C_4}_{C_2}"'] \\
\underline{M}(C_8/C_2) \arrow[u, bend right, "tr^{C_4}_{C_2}"'] \arrow[d, bend right, "res^{C_2}_{e}"'] \\
\underline{M}(C_8/e) \arrow[u, bend right, "tr^{C_2}_{e}"']
\end{tikzcd}$$

Figure 1.1: A C_8-Mackey Functor

Remark 1.1.1. Definition 1.1.1 will hold for any finite group G, and if G is non-abelian then we can create a definition similar to Definition 1.1.2 for a G-Mackey functor. However, when the group is non-abelian a map $G/K \to G/H$ exists when K is subconjugate to H and $res^G_H tr^G_H(x)$ becomes a much more complicated sum over double cosets (see [16]).

1.1.1 Examples

Example 1.1.2. *Fixed Point Mackey Functors.* The most basic type of Mackey functor is the fixed point Mackey functor. Let D be a group equipped with an action of G. We will denote the fixed point Mackey functor of D by $\underline{D}$, and for all subgroups G of K we define $\underline{D}(G/H)$ by

$$\begin{aligned}\underline{D}(G/H) &= D^H \\ &= \{d \in D | h \cdot d = d \text{ for all } h \in H\} \\ &= \text{The subgroup of } H\text{-fixed points in } D.\end{aligned}$$

For all subgroups K of H, the restriction map $res^H_K : D^H \to D^K$ is simply inclusion of fixed points, and we define the transfer map $tr^H_K : D^K \to D^H$ by

$$tr^H_K(d) = \sum_{\gamma \in W_H(K)} \gamma \cdot d.$$

The most common fixed point Mackey functor is the constant Mackey functor $\underline{\mathbb{Z}}$. Consider $\mathbb{Z}$ as a group with trivial G-action. Then $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$ for all subgroups H in G, every restriction map is the identity, and for all x in $\underline{\mathbb{Z}}(G/K)$, $tr^H_K(x) = |H/K|x$ whenever K is a subgroup of H. We give the Mackey functor diagram for the C_{p^2}-

Constant Mackey functor below.

$$\begin{tikzcd}
\underline{\mathbb{Z}}(C_{p^2}/C_{p^2}) = \mathbb{Z} \arrow[d, bend right, "res_{C_p}^{C_{p^2}} = id"'] \\
\underline{\mathbb{Z}}(C_{p^2}/C_p) = \mathbb{Z} \arrow[u, bend right, "tr_{C_p}^{C_{p^2}} = (\times p)"'] \arrow[d, bend right, "res_e^{C_p} = id"'] \\
\underline{\mathbb{Z}}(C_{p^2}/e) = \mathbb{Z} \arrow[u, bend right, "tr_e^{C_p} = (\times p)"']
\end{tikzcd}$$

We can also construct the fixed point C_2-Mackey functor of $\mathbb{Z}\{C_2\}$, the free abelian group on the set of elements of C_2. This group consists of elements of the form $a + b\gamma$ where γ is the non-zero element of C_2, and C_2 acts by $\gamma \cdot (a + b\gamma) = b + a\gamma$. Moreover, an element is fixed by C_2 if and only if it is of the form $a + a\gamma$. Thus, $\underline{\mathbb{Z}\{C_2\}}(C_2/e) = \mathbb{Z}\{C_2\}$, and $\underline{\mathbb{Z}\{C_2\}}(C_2/C_2)$ consists of a copy of $\mathbb{Z}$ generated by $1 + \gamma$, denoted $\mathbb{Z}\{1 + \gamma\}$. Then

$$res_e^{C_2}(a) = a + a\gamma$$

and

$$tr_e^{C_2}(a + b\gamma) = \sum_{\gamma \in C_2} \gamma \cdot (a + b\gamma) = (a + b) + (a + b)\gamma.$$

We can describe $\underline{\mathbb{Z}\{C_2\}}$ using the Mackey functor diagram shown below.

$$
\begin{tikzcd}
a \arrow[d, mapsto] & \underline{\mathbb{Z}\{C_2\}}(C_2/C_2) = \mathbb{Z}\{1+\gamma\} \arrow[d, bend right, "res_e^{C_2}"'] & (a+b)+(a+b)\gamma \\
a + a\gamma & \underline{\mathbb{Z}\{C_2\}}(C_2/e) = \mathbb{Z}\{C_2\} \arrow[u, bend right, "tr_e^{C_2}"'] & a+b\gamma \arrow[u, mapsto]
\end{tikzcd}
$$

Example 1.1.3. *The Burnside Mackey functor, $\underline{A}$.* The Burnside Mackey functor is the most important Mackey functor. Not only is every Mackey functor a module over $\underline{A}$, but also if $\mathbb{S}^0$ is the sphere spectrum then $\underline{\pi_0}(\mathbb{S}^0) = \underline{A}$ [14].

For all subgroups H of G we define $\underline{A}(G/H)$ to be the Grothendieck group on $\mathcal{F}in(H)$, the set of isomorphism classes of finite H-sets, and thus,

$$\underline{A}(G/H) = \mathbb{Z}\{\mathcal{F}in(H)\}/_{[X \amalg Y]=[X]+[Y]}.$$

Further, for all subgroups K of H the transfer map $tr_K^H : \underline{A}(G/K) \to \underline{A}(G/H)$ is given by $tr_K^H([Y]) = [H \times_K Y]$ and $res_K^H : \underline{A}(G/H) \to \underline{A}(G/K)$ by $res_K^H([X]) = [i_K^* X]$ where $i_K^* : \mathscr{S}et_H^{Fin} \to \mathscr{S}et_K^{Fin}$ is the restriction functor that sends an H-set to its underlying K-set. The Weyl action is trivial.

More specifically, we will construct a Mackey functor diagram for the C_p-Burnside Mackey functor. Because there are only two C_p-orbits, C_p/C_p and C_p/e, we only need to determine $\underline{A}(C_p/e)$, $\underline{A}(C_p/C_p)$, $tr_e^{C_p} : \underline{A}(C_p/e) \to \underline{A}(C_p/C_p)$ and $res_e^{C_p} : \underline{A}(C_p/C_p) \to \underline{A}(C_p/e)$. When H is the trivial subgroup $\mathcal{F}in(H)$ consists only of the

isomorphism class of the single point set $[e/e]$. Hence,

$$\underline{A}(C_p/e) = \mathbb{Z}\{[e/e]\} = \mathbb{Z}.$$

There are two isomorphism classes of finite C_p-sets, $[C_p/C_p]$ and $[C_p/e]$, and so

$$\underline{A}(C_p/C_p) = \mathbb{Z}\{[C_p/C_p]\} \oplus \mathbb{Z}\{[C_p/e]\} = \mathbb{Z} \oplus \mathbb{Z}.$$

Then

$$tr_e^{C_p}([e/e]) = [C_p \times_e e/e] = [C_p/e] = (0,1),$$

$$res_e^{C_p}([C_p/C_p]) = [e/e] = 1, \text{ and}$$

$$res_e^{C_p}([C_p/e]) = [\underbrace{e/e \amalg \cdots \amalg e/e}_{\text{p times}}] = p[e/e] = p.$$

We now give the C_p-Burnside Mackey functor diagram.

$$\begin{tikzcd}
(1,0) \arrow[d, mapsto] & (0,1) \arrow[d, mapsto] & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, bend right, "res_e^{C_p}"'] & (0,1) \\
1 & p & \mathbb{Z} \arrow[u, bend right, "tr_e^{C_p}"'] & 1 \arrow[u, mapsto]
\end{tikzcd}$$

1.2 The Category of G-Mackey Functors

Let $\mathcal{M}ack_G$ be the category of G-Mackey functors. A morphism $\phi : \underline{M} \to \underline{L}$ in $\mathcal{M}ack_G$ consists of a collection of group homomorphisms $\phi_H : \underline{M}(G/H) \to \underline{L}(G/H)$ for all subgroups H of G such that each ϕ_H is $W_G(H)$-equivariant and whenever K is a subgroup of H $res_K^H \phi_H = \phi_K res_K^H$ and $tr_K^H \phi_K = \phi_H tr_K^H$. We can visualize ϕ using

the Mackey functor diagrams of $\underline{M}$ and $\underline{L}$. For example, if $\underline{M}$ and $\underline{L}$ are C_p-Mackey functors then we can describe ϕ with the following diagram.

$$\begin{tikzcd}
\underline{M}(C_p/C_p) \arrow[rr, "\phi_{C_p}"] \arrow[d, bend right, "res_e^{C_p}"'] & & \underline{L}(C_p/C_p) \arrow[d, bend right, "res_e^{C_p}"'] \\
\underline{M}(C_p/e) \arrow[rr, "\phi_e"] \arrow[u, bend right, "tr_e^{C_p}"'] & & \underline{L}(C_p/e) \arrow[u, bend right, "tr_e^{C_p}"']
\end{tikzcd}$$

1.2.1 The Box Product

The category of G-Mackey functors has a symmetric monoidal product called the *box product* and denoted $\square$. Given any finite group G we can define the box product of $\mathcal{M}ack_G$ in terms of a double coend or left Kan extension [11]. However, we will provide a detailed description of Gaunce Lewis' constructive definition for C_{p^n}-Mackey functors [6]. More specifically, given C_{p^n}-Mackey functors $\underline{M}$ and $\underline{L}$, we will build a Mackey functor diagram for $\underline{M} \square \underline{L}$.

The box product is the Mackey functor analogue of the tensor product, and so we might hope that $(\underline{M} \square \underline{L})(G/H)$ is simply $\underline{M}(G/H) \otimes \underline{L}(G/H)$, but alas this definition does not support the transfer map. Thus, instead, we need Definition 1.2.1.

Definition 1.2.1. Given C_{p^n}-Mackey functors $\underline{M}$ and $\underline{L}$ we inductively define $\underline{M} \square \underline{L}$ as follows. For all subgroups H of G define

$$(\underline{M} \square \underline{L})(G/H) := \big(\underline{M}(G/H) \otimes \underline{L}(G/H) \oplus (\underline{M} \square \underline{L})(G/K)/_{W_H(K)}\big)/_{FR}.$$

1. The subgroup K is the maximal subgroup of H, and the transfer map tr_K^H is the quotient map onto the $(\underline{M} \,\Box\, \underline{L})(G/K)/_{W_H(K)}$ summand. We will refer to $(\underline{M} \,\Box\, \underline{L})(G/K)/_{W_H(K)}$ as $Im(tr_K^H)$ and an element in this summand as $tr_K^H(x)$.

2. The submodule FR is called the *Frobenius reciprocity* submodule and is generated by elements of the form

$$a \otimes tr_{K'}^H(b) - tr_K^H tr_{K'}^K(res_{K'}^H(a) \otimes b)$$

and

$$tr_{K'}^H(c) \otimes d - tr_K^H tr_{K'}^K(c \otimes res_{K'}^H(d))$$

for all subgroups K' of H, and all elements a in $\underline{M}(G/H)$, b in $\underline{L}(G/K')$, c in $\underline{M}(G/K')$, and d in $\underline{L}(G/H)$.

3. The Weyl action is given by $\gamma \cdot (a \otimes b) = \gamma a \otimes \gamma b$ for all γ in $W_G(H)$.

4. We define the restriction map $res_K^H : (\underline{M} \,\Box\, \underline{L})(G/H) \to (\underline{M} \,\Box\, \underline{L})(G/K)$ by $res_K^H(a) \otimes res_K^H(b)$ for $a \otimes b$ in $\underline{M}(G/H) \otimes \underline{L}(G/H)$ and

$$res_K^H(tr_K^H(x)) = \sum_{\gamma \in W_H(K)} \gamma \cdot tr_K^H(x)$$

for all $tr_K^H(x)$ in $Im(tr_K^H)$.

Example 1.2.1. Let $\underline{M}$ and $\underline{L}$ be C_4-Mackey functors. We display the Mackey functor $\underline{M} \,\Box\, \underline{L}$ in Figure 1.2.

$$\begin{tikzcd}
\left[\underline{M}(C_4/C_4) \otimes \underline{L}(C_4/C_4) \oplus \overbrace{(\underline{M} \,\Box\, \underline{L})(C_4/C_2)/_{W_{C_4}(C_2)}}^{Im(tr_{C_2}^{C_4})}\right]/_{FR} \arrow[d, bend right, "res_{C_2}^{C_4}"'] \\
\left[\underline{M}(C_4/C_2) \otimes \underline{L}(C_4/C_2) \oplus \overbrace{(\underline{M}(C_4/e) \otimes \underline{L}(C_4/e))/_{C_2}}^{Im(tr_e^{C_2})}\right]/_{FR} \arrow[u, bend right, "tr_{C_2}^{C_4}"'] \arrow[d, bend right, "res_e^{C_2}"'] \\
\underline{M}(C_4/e) \otimes \underline{L}(C_4/e) \arrow[u, bend right, "tr_e^{C_2}"']
\end{tikzcd}$$

Figure 1.2: The Box Product of C_4-Mackey Functors

Remark 1.2.2. We can extend this definition of the box product to an i-fold box product of C_{p^n}-Mackey functors by defining

$$(\underline{M}_1 \,\Box\, \underline{M}_2 \,\Box \dots \Box\, \underline{M}_i)(G/H) = \big(\underline{M}_1(G/H) \otimes \cdots \otimes \underline{M}_i(G/H) \oplus Im(tr_K^H)\big)/_{FR}.$$

The submodule $Im(tr_K^H) = (\underline{M}_1 \,\Box \cdots \Box\, \underline{M}_i)(G/K)/_{W_H(K)}$ where K is the maximal subgroup of H, and the Frobenius reciprocity submodule is generated by elements of the form

$$\begin{aligned} &m_1 \otimes m_2 \otimes \cdots \otimes tr_{K'}^H(b)_j \otimes \cdots \otimes m_i - \\ &\quad tr_K^H tr_{K'}^K(res_{K'}^H(m_1) \otimes res_{K'}^H(m_2) \otimes \cdots \otimes b_j \otimes \cdots \otimes res_{K'}^H(m_i)) \end{aligned}$$

for all $1 \leq j \leq i$.

Remark 1.2.3. A map of Mackey functors $\psi : \underline{M} \,\Box\, \underline{L} \to \underline{P}$ determines and is

determined by a collection of group homomorphisms

$$\theta_H : \underline{M}(G/H) \otimes \underline{L}(G/H) \to \underline{P}(G/H)$$

for all subgroups H of G such that the diagrams below commute whenever K is a subgroup of H [11].

$$\begin{tikzcd}
\underline{M}(G/H) \otimes \underline{L}(G/H) \arrow[r, "\theta_H"] \arrow[d, "res^H_K \otimes res^H_K"'] & \underline{P}(G/H) \arrow[d, "res^H_K"] \\
\underline{M}(G/K) \otimes \underline{L}(G/K) \arrow[r, "\theta_K"] & \underline{P}(G/K)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underline{M}(G/H) \otimes \underline{L}(G/H) \arrow[r, "\theta_H"] & \underline{P}(G/H) \\
\underline{M}(G/K) \otimes \underline{L}(G/H) \arrow[ur, "tr^H_K \otimes id"] \arrow[dr, "id \otimes res^H_K"'] & & \\
 & \underline{M}(G/K) \otimes \underline{L}(G/K) \arrow[r, "\theta_K"] & \underline{P}(G/K) \arrow[uu, "tr^H_K"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \underline{M}(G/H) \otimes \underline{L}(G/H) \arrow[r, "\theta_H"] & \underline{P}(G/H) \\
\underline{M}(G/H) \otimes \underline{L}(G/K) \arrow[ur, "id \otimes tr^H_K"] \arrow[dr, "res^H_K \otimes id"'] & & \\
 & \underline{M}(G/K) \otimes \underline{L}(G/K) \arrow[r, "\theta_K"] & \underline{P}(G/K) \arrow[uu, "tr^H_K"']
\end{tikzcd}$$

Furthermore, a map $\Psi : \underline{M}_1 \,\square\, \underline{M}_2 \,\square \ldots \square\, \underline{M}_i \to \underline{P}$ determines and is determined by a collection of maps

$$\Theta_H : \underline{M}_1(G/H) \otimes \underline{M}_2(G/H) \otimes \cdots \otimes \underline{M}_i(G/H) \to \underline{P}(G/H)$$

for all subgroups H of G such that the i-fold analogues of the above diagrams commute whenever K is a subgroup of H.

1.3 Green Functors

The unit for the box product is the Burnside Mackey functor, and thus $(\mathcal{M}ack_G, \square, \underline{A})$ forms a symmetric monoidal category. The monoidal objects are *G-Green functors*.

We will provide two definitions of Green functors. The first is the category theoretic definition given by Lewis in [7]. The second is a constructive definition similar to Definition 1.1.2 of a Mackey functor. Ventura shows that these definitions are equivalent in [15].

Definition 1.3.1. A *G-Green functor* $\underline{R}$ (or just *Green functor* when the group is clear) is a G-Mackey functor together with maps $m : \underline{R} \square \underline{R} \to \underline{R}$ and $1_{\underline{R}} : \underline{A} \to \underline{R}$ such that the diagrams below commute.

$$\begin{tikzcd}
\underline{R} \square \underline{R} \square \underline{R} \arrow[r, "id \square m"] \arrow[d, "m \square id"'] & \underline{R} \square \underline{R} \arrow[d, "m"] \\
\underline{R} \square \underline{R} \arrow[r, "m"] & \underline{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{A} \square \underline{R} \arrow[r, "1_{\underline{R}} \square id"] \arrow[dr, "\cong"'] & \underline{R} \square \underline{R} \arrow[d, phantom] & \underline{A} \square \underline{R} \arrow[l, "id \square 1_{\underline{R}}"'] \arrow[dl, "\cong"] \\
 & \underline{R} &
\end{tikzcd}$$

Furthermore, a Green functor $\underline{R}$ is *commutative* if the diagram below commutes where the map τ permutes the coordinates of $\underline{R} \square \underline{R}$.

$$\begin{tikzcd}
\underline{R} \square \underline{R} \arrow[rr, "\tau"] \arrow[dr, "m"'] & & \underline{R} \square \underline{R} \arrow[dl, "m"] \\
 & \underline{R} &
\end{tikzcd}$$

Definition 1.3.2. A Mackey functor $\underline{R}$ is a *Green functor* if

- $\underline{R}(G/H)$ is a ring for all orbits G/H,
- all restriction maps $res^H_K : \underline{R}(G/H) \to \underline{R}(G/K)$ are (unit-preserving) ring homomorphisms, and
- $\underline{R}$ satisfies *Frobenius reciprocity*: If K is a subgroup of H then

$$tr^H_K(a) \cdot b = tr^H_K(a \cdot res^H_K(b))$$

for all a in $\underline{R}(G/K)$ and b in $\underline{R}(G/H)$

Moreover, a Green functor $\underline{R}$ is *commutative* if every $\underline{R}(G/H)$ is a *commutative* ring.

As with Mackey functors, we will often visualize a Green functor diagrammatically. A morphism $\phi : \underline{M} \to \underline{L}$ of Mackey functors is a morphism of Green functors if all group homomorphisms $\phi_H : \underline{M}(G/H) \to \underline{L}(G/H)$ are (unit-preserving) ring homomorphisms.

1.3.1 Examples

Example 1.3.1. A fixed point Mackey functor $\underline{D}$ will extend to a fixed point Green functor if we can endow the group D with a ring structure that is compatible with the G-action. In particular we require that $\gamma \cdot (ab) = (\gamma a) \cdot (\gamma b)$ and $\gamma 1 = 1$ for all γ in G and a and b in D. For example, the constant Mackey functor $\underline{\mathbb{Z}}$ inherits the structure of a Green functor. However, $\underline{\mathbb{Z}\{C_2\}}$ does not. We can endow $\mathbb{Z}\{C_2\}$ with a ring structure by letting $(a + b\gamma)(c + d\gamma)$ equal $(ac + bd) + (ad + bc)\gamma$, but this multiplication is not compatible with the C_2-action.

Example 1.3.2. The Burnside Mackey functor naturally extends to a Green functor. We can define $\underline{A}(G/H)$ to be the Grothendieck ring on $\mathcal{F}in(H)$. Hence, if $[X]$ and $[Y]$ are isomorphism classes of finite H-sets then $[X][Y] = [X \times Y]$, and the multiplicative unit is the isomorphism class of the single point set $[H/H]$. Further, the restriction maps are ring homomorphisms because i_K^* is a forgetful functor for all subgroups K,

and thus preserves products. Frobenius reciprocity is satisfied because

$$[(H \times_K X) \times Y] = [H \times_K (X \times i_K^* Y)].$$

Indeed, an isomorphism $H \times_K (X \times i_K^* Y) \to (H \times_K X) \times Y$ is given by

$$(h, x, y) \mapsto (h, x, hy).$$

We can now determine the ring structure of $\underline{A}(C_p/C_p)$. The unit is the isomorphism class $[C_p/C_p]$, and

$$[C_p/e][C_p/e] = [C_p/e \times C_p/e] = \underbrace{[C_p/e \amalg \cdots \amalg C_p/e]}_{p \text{ times}} = p[C_p/e].$$

Thus, if we let t be $[C_p/e]$ it follows that $\underline{A}(C_p/C_p) = \mathbb{Z}[t]/_{t^2=pt}$, $tr(1) = t$, and $res(t) = p$.

$$\begin{tikzcd} 1 \arrow[d, mapsto] & t \arrow[d, mapsto] & \mathbb{Z}[t]/_{t^2=pt} \arrow[d, bend right, "res_e^{C_p}"'] & t \\ 1 & p & \mathbb{Z} \arrow[u, bend right, "tr_e^{C_p}"'] & 1 \arrow[u, mapsto] \end{tikzcd}$$

1.4 Tambara Functors

Tambara functors are, in essence, commutative Green functors with extra structure in the form of a third map induced from a map between orbits. This map is the multiplicative analogue of the transfer and is called the *norm* map.

We will provide two equivalent definitions of a Tambara functor. The first is Tambara's original definition and is similar to Definition 1.1.1 of a Mackey functor [13]. The second definition is a more constructive definition reminiscent of Definitions 1.1.2 and 1.3.2. However, before stating Tambara's definition we need to introduce some extra terminology.

Let $f : X \to Y$ be a morphism in $\mathscr{S}et_G^{Fin}$ and for X in $\mathscr{S}et_G^{Fin}$ let $\mathscr{S}et_G^{Fin}|X$ to be the category of finite G-sets over X. Then the pullback functor associated to $f : X \to Y$ is given by

$$\mathscr{S}et_G^{Fin}|Y \to \mathscr{S}et_G^{Fin}|X$$

$$(B \to Y) \mapsto (X \times_Y B \to X)$$

and has right adjoint

$$\mathscr{S}et_G^{Fin}|X \to \mathscr{S}et_G^{Fin}|Y$$

$$(A \xrightarrow{q} X) \mapsto (\prod_f A \xrightarrow{\prod_f q} Y)$$

We define $\prod_f A \xrightarrow{\prod_f q} Y$ as follows. The G-set $\prod_f A$ consists of all maps $s : f^{-1}(y) \to A$ such that $q \circ s(x) = x$ for all x in $f^{-1}(y)$ and whose domain is the preimage $f^{-1}(y)$ in X of any y in Y. The G-action is $(\gamma \cdot s)(x) = \gamma s(\gamma^{-1}x)$ for all γ in G. We then define the map $\prod_f q : \prod_f A \to Y$ by $(\prod_f q)(s) = y$ if the domain of s is $f^{-1}(y)$.

Moreover, we can form the pullback $X \times_Y \prod_f A$, since

$$\begin{aligned} X \times_Y \prod_f A &= \{(x,s) | f(x) = (\prod_f q)(s)\} \\ &= \{(x,s) | s : f^{-1}(f(x)) \to A \text{ is such that } q \circ s(z) = z \\ &\quad \text{for all } z \in f^{-1}(f(x))\}. \end{aligned}$$

This pullback fits into a commutative diagram shown below where the map p is the projection map and e is the evaluation map $(x,s) \mapsto s(x)$.

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "q"'] & X \times_Y \prod_f A \arrow[l, "e"'] \arrow[d, "p"] \\ Y & & \prod_f A \arrow[ll, "\prod_f q"'] \end{tikzcd}$$

We call a diagram that is isomorphic to this diagram an *exponential diagram* [13].

Definition 1.4.1. A *G-Tambara functor* $\underline{S}$ is a map on $\mathscr{S}et_G^{Fin}$ that sends each finite G-set X to a commutative ring $\underline{S}(X)$ and that assigns to each map $f : X \to Y$ in $\mathscr{S}et_G^{Fin}$ three maps $f^* : \underline{S}(Y) \to \underline{S}(X)$, $f_* : \underline{S}(X) \to \underline{S}(Y)$, and $f_\star : \underline{S}(X) \to \underline{S}(Y)$ such that

1. $\underline{S}$ converts a disjoint union of finite G-sets to a direct sum of commutative rings,

2. f^* is a ring homomorphism, f_* is a homomorphism of additive monoids, and $f_\star$ is a homomorphism of multiplicative monoids,

3. if $f : X \to Y$ and $g : Y \to Z$ are in $\mathscr{S}et_G^{Fin}$, then $(gf)^* = f^*g^*$, $(gf)_* = g_*f_*$, $(gf)_\star = g_\star f_\star$ and $(1_X)^* = (1_X)_* = (1_X)_\star = 1_{\underline{S}(X)}$,

4. for a pullback diagram of finite G-sets

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd}$$

the following induced diagrams commute

$$\begin{tikzcd} \underline{S}(X) \arrow[r, "f_*"] & \underline{S}(Y) \\ \underline{S}(Z) \arrow[u, "g^*"] \arrow[r, "k_*"] & \underline{S}(W) \arrow[u, "h^*"'] \end{tikzcd} \qquad \begin{tikzcd} \underline{S}(X) \arrow[r, "f_\star"] & \underline{S}(Y) \\ \underline{S}(Z) \arrow[u, "g^*"] \arrow[r, "k_\star"] & \underline{S}(W) \arrow[u, "h^*"'] \end{tikzcd}$$

5. for an exponential diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Z \arrow[l, "g"'] & X' \arrow[l, "e"'] \arrow[d, "f'"] \\ Y & & Y' \arrow[ll, "h"'] \end{tikzcd}$$

the induced diagram below commutes.

$$\begin{tikzcd} \underline{S}(X) \arrow[d, "f_\star"'] & \underline{S}(Z) \arrow[l, "g_*"'] \arrow[r, "e^*"] & \underline{S}(X') \arrow[d, "f'_\star"] \\ \underline{S}(Y) & & \underline{S}(Y') \arrow[ll, "h_*"'] \end{tikzcd}$$

Given a map $f : X \to Y$ of G-sets the maps f_* and f^* are the induced transfer and restriction maps that arise in the definition of a Mackey functor. The third induced map $f_\star$ is the norm map.

As with Mackey functors we can completely understand a Tambara functor $\underline{S}$ once we determine $\underline{S}(G/H)$ for all orbits G/H and the restriction, transfer, and norm maps induced from all maps $G/K \to G/H$ between orbits. The induced restriction and transfer maps are still denoted res_K^H and tr_K^H, and the induced norm map is denoted

N_K^H. The Weyl groups $W_H(K)$ still act on $\underline{M}(G/K)$, and Property 3 of Definition 1.4.1 requires that $res_K^H tr_K^H(x) = \sum_{\gamma \in W_H(K)} \gamma \cdot x$ and $res_K^H N_K^H(x) = \prod_{\gamma \in W_H(K)} \gamma \cdot x$. Moreover, the norm maps are not additive, but Property 4 provides a formula for the norm of a sum since we can build the following exponential diagram where the map ∇ is the canonical fold map.

$$\begin{tikzcd}
G/K \arrow[d, "f"'] & G/K \amalg G/K \arrow[l, "\nabla"'] & G/K \times_{G/H} \prod_f (G/K \amalg G/K) \arrow[l, "e"'] \arrow[d, "p"] \\
G/H & & \prod_f (G/K \amalg G/K) \arrow[ll, "\prod_f \nabla"']
\end{tikzcd}$$

This diagram induces the diagram below, which gives a description of $N_K^H(a+b)$ in terms of $N_K^H(a)$, $N_K^H(b)$ and transfer terms.

$$\begin{tikzcd}
\underline{S}(G/K) \arrow[d, "N_K^H"'] & \underline{S}(G/K) \oplus \underline{S}(G/K) \arrow[l, "\nabla_*"'] \arrow[r, "e^*"] & \underline{S}(G/K \times_{G/H} \prod_f (G/K \amalg G/K)) \arrow[d, "p_\star"] \\
\underline{S}(G/H) & & \underline{S}(\prod_f (G/K \amalg G/K)) \arrow[ll, "(\prod_f \nabla)_*"']
\end{tikzcd}$$

We can also derive a description of the norm of a transfer using Property 4. More specifically, the exponential diagram

$$\begin{tikzcd}
G/K \arrow[d, "f"'] & G/K' \arrow[l, "q"'] & G/K \times_{G/H} \prod_f G/K' \arrow[l, "e"'] \arrow[d, "p"] \\
G/H & & \prod_f G/K' \arrow[ll, "\prod_f q"']
\end{tikzcd}$$

induces the diagram

$$\begin{tikzcd}
\underline{S}(G/K) \arrow[d, "N_K^H"'] & \underline{S}(G/K') \arrow[l, "tr_{K'}^K"'] \arrow[r, "e^*"] & \underline{S}(G/K \times_{G/H} \prod_f G/K') \arrow[d, "p_\star"] \\
\underline{S}(G/H) & & \underline{S}(\prod_f G/K') \arrow[ll, "(\prod_f q)_*"']
\end{tikzcd}$$

1.4.1 The Norm of a Sum in a C_{p^n}-Tambara Functor

In essence, these exponential diagrams tell us that applying the norm map is like taking powers. Thus, whilst the norm is not additive it is governed by universal polynomials, and if the group G is not too complicated then we can determine the norm of a sum using these polynomials. More specifically, norm maps must behave like multiplying over the Weyl action, and so if $G = C_{p^n}$, γ is the generator of G, and H is a subgroup of G then we can describe $N_H^G(a+b)$ by examining

$$\prod_{\gamma^t \in W_G(H)} \gamma^t \cdot (a+b) \quad = \quad (a+b)(\gamma a + \gamma b) \cdots (\gamma^{|G/H|-1} a + \gamma^{|G/H|-1} b). \quad (1.4.1)$$

The expansion of this product consists of a term of the form $a\gamma a \cdots \gamma^{|G/H|-1} a$, a term of the form $b\gamma b \cdots \gamma^{|G/H|-1} b$, and $\sum_{k=1}^{|G/H|-1} \binom{|G/H|}{k}$-many mixed terms that are products of a, b, and their Weyl conjugates. The number of mixed terms should not be alarming since

$\prod_{\gamma \in W_G(H)} \gamma(a+b)$ will have the same number of terms as the polynomial expansion of $(a+b)^{|G/H|}$. We stress that no $\gamma^t a$ or $\gamma^t b$ for any $0 \leq t \leq |G/H|$ occurs more than once in any mixed term. If $\gamma^t a^m$ or $\gamma^t b^m$ appears in a mixed term then $m = 1$. Moreover, we denote the expansion of Equation 1.4.1 by $\prod \gamma^t(a+b)$. This expansion is universally determined by the group G and depends neither on the given Tambara functor nor on a and b.

We can analyze $\prod \gamma^t(a+b)$ even further to develop a thorough description of $N_H^G(a+b)$. First, the terms $a\gamma a \cdots \gamma^{|G/H|-1} a$ and $b\gamma b \cdots \gamma^{|G/H|-1} b$ translate to $N_H^G(a)$

and $N_H^G(b)$, respectively, and thus we will see $N_H^G(a)$ and $N_H^G(b)$ in the expansion of $N_H^G(a+b)$. We can then group the mixed terms of $\prod \gamma^t(a+b)$ into sums over orbits. If we let $(\vec{ab})^H$ denote a mixed term whose stabilizer subgroup is H then all of the $W_G(H)$-conjugates of $(\vec{ab})^H$ are also mixed terms in $\prod \gamma^t(a+b)$. Thus, the expansion includes the summation $\sum_{\gamma^t \in W_G(H)} \gamma^t \cdot (\vec{ab})^H$. Since the transfer map is analogous to summing over the Weyl action this summation becomes $tr_H^G((\vec{ab})^H)$ in the description of $N_H^G(a+b)$. If there are i-many orbits of mixed terms stabilized by H in $\prod \gamma^t(a+b)$ then all such mixed terms translate to $tr_H^G(\sum_{j=1}^i (\vec{ab})_j^H)$ in $N_H^G(a+b)$ where each $(\vec{ab})_j^H$ is a representative from the j^{th} $W_G(H)$-orbit. Further, the summation $\sum_{j=1}^i (\vec{ab})_j^H$ is a polynomial in a, b, and their $W_G(H)$-conjugates and is universally determined by the group G. Hence, we denote this summation $g_H(a,b)$.

If a mixed term has stabilizer subgroup K such that $H < K < G$ then we can write this term as $\prod_{\gamma^s \in W_K(H)} \gamma^s \cdot (\vec{ab})^K$ where $(\vec{ab})^K$ is a product of a, b, and some of their $W_G(K)$-conjugates. Again, if $\gamma^s a^m$ or $\gamma^s b^m$ appear in $(\vec{ab})^K$ then $m = 1$. Moreover, if $\prod_{\gamma^s \in W_K(H)} \gamma^s \cdot (\vec{ab})^K$ is a term in $\prod \gamma^t(a+b)$ then so is every element in its $W_G(K)$-orbit. In other words,

$$\sum_{\gamma^t \in W_G(K)} \gamma^t \cdot \left(\prod_{\gamma^s \in W_K(H)} \gamma^s \cdot (\vec{ab})^K \right)$$

is a part of the expansion. But, since we can regard $\prod_{\gamma^s \in W_K(H)} \gamma^s \cdot x$ as $N_K^H(x)$ and $\sum_{\gamma^t \in W_G(K)} \gamma^t \cdot y$ as $tr_K^G(y)$ it follows that

$$\sum_{\gamma^t \in W_G(K)} \gamma^t \cdot \left(\prod_{\gamma^t \in W_K(H)} \gamma^t \cdot (\vec{ab})^K \right)$$

translates to

$$tr_K^G(N_H^K((\vec{ab})^K))$$

in the description of $N_H^G(a+b)$. If there are i'-many $W_G(K)$-orbits in $\prod \gamma(a+b)$ then the sum of all mixed terms stabilized by K can be written as

$$tr_K^G\left(\sum_{j=1}^{i'} N_H^K((\vec{ab})_j^K)\right)$$

where each $N_H^K((\vec{ab})_j^K)$ is a representative from the j^{th} $W_G(K)$-orbit. We should view the summation $\sum_{j=1}^{i'} N_H^K((\vec{ab})_j^K)$ as a polynomial in a, b, and their $W_G(K)$-conjugates. We stress that this polynomial is universally determined by G.

Therefore, even though we do not have an explicit formula for $N_H^G(a+b)$ we can conclude that for all subgroups H of G

$$N_H^G(a+b) = \qquad (1.4.2)$$
$$N_H^G(a) + N_H^G(b) + tr_H^G(g_H(a,b)) + \sum_{H<K<G} tr_K^G\left(\sum_{j=1}^{i'} N_H^K((\vec{ab})_j^K)\right).$$

When H is maximal in G Equation 1.4.2 simplifies to

$$N_H^G(a+b) \quad = \quad N_H^G(a) + N_H^G(b) + tr_H^G(g_H(a,b)). \qquad (1.4.3)$$

The polynomial $g_H(a,b)$ is a polynomial in a, b, and their $W_G(H)$-conjugates, and $\sum_{j=1}^{i'} N_H^K((\vec{ab})_j^K)$ is a polynomial in a, b, and their $W_G(K)$-conjugates. These polynomials are universally determined by the group G and consist only of monomials that are products of single power terms.

Example 1.4.1. If the group G is small enough we can use Equation 1.4.2 or Equation 1.4.3 to find an explicit formula for $N_H^G(a+b)$. For example, if γ is the generator of C_2, then we know that for any C_2-Tambara functor

$$N_e^{C_2}(a+b) = N_e^{C_2}(a) + N_e^{C_2}(b) + tr_e^{C_2}\left(g_e(a,b)\right)$$

where $g_e(a,b)$ is a polynomial with variables a, b, γa, and γb and is universally determined by C_2. However, we can explicitly describe this polynomial by computing

$$\begin{aligned}\prod_{\gamma\in C_2} \gamma\cdot(a+b) &= (a+b)(\gamma a+\gamma b)\\ &= a\gamma a + a\gamma b + b\gamma a + b\gamma b\\ &= a\gamma a + \sum_{\gamma\in C_2}\gamma\cdot(a\gamma b) + b\gamma b\end{aligned}$$

Thus, $g_e(a,b) = a\gamma b$, and for any C_2-Tambara functor, an explicit formula for $N_e^{C_2}(a+b)$ is given by

$$N_e^{C_2}(a+b) = N_e^{C_2}(a) + N_e^{C_2}(b) + tr_e^{C_2}(a\gamma b).$$

Example 1.4.2. Similarly, we can determine an explicit formula for $N_e^{C_4}(a+b)$ in any C_4-Tambara functor. Letting γ be the generator of C_4, the norm of a sum in a C_4-Tambara functor is determined by

$$\prod_{\gamma\in C_4}\gamma\cdot(a+b) = (a+b)(\gamma a+\gamma b)(\gamma^2 a+\gamma^2)(\gamma^3 a+\gamma^3 b).$$

Expanding the above expression results in a 16 term summation that translates to

$$N_e^{C_4}(a+b) =$$

$$N_e^{C_4}(a) + N_e^{C_4}(b) + tr_{C_2}^{C_4}(N_e^{C_2}(a\gamma b)) + tr_e^{C_4}(a\gamma a\gamma^2 a\gamma^3 b + b\gamma b\gamma^2 b\gamma^3 a + a\gamma b\gamma^2 b\gamma^3 a)$$

Further, the above discussion describes $N_e^{C_4}(a+b)$ by

$$N_e^{C_4}(a+b) = N_e^{C_4}(a) + N_e^{C_4}(b) + tr_{C_2}^{C_4}(\sum_j N_e^{C_2}((\vec{ab})_j^{C_2})) + tr_e^{C_4}(g_e(a,b)).$$

Hence, by computing $\prod_{\gamma \in C_4} \gamma \cdot (a+b)$ we have discovered that for every C_4-Tambara functor the polynomial

$$\sum_j N_e^{C_2}((\vec{ab})_j^{C_2}) = N_e^{C_2}(a\gamma b)$$

and

$$g_e(a,b) = a\gamma a\gamma^2 a\gamma^3 b + b\gamma b\gamma^2 b\gamma^3 a + a\gamma b\gamma^2 b\gamma^3 a.$$

1.4.2 The Norm of a Transfer in a C_{p^n}-Tambara Functor

Let H and K be subgroups of G such that $K < H$. We can also use the exponential diagrams to realize $N_H^G(tr_K^H(x))$ as

$$\prod_{\gamma \in W_G(H)} \gamma \cdot \left(\sum_{\delta \in W_H(K)} \delta \cdot x \right).$$

Therefore, even though we will not be able to find a closed formula for $N_H^G(tr_K^H(x))$, letting H be C_{p^k}, K be C_{p^i}, and η be p^{n-k} we can obtain a better understanding of $N_H^G(tr_K^H(x))$ by examining

$$\prod_{\gamma^t \in W_G(H)} \gamma^t \cdot \left(\sum_{\gamma^{j\eta} \in W_H(K)} \gamma^{j\eta} \cdot x \right).$$

First, the summation

$$\sum_{\gamma^{j\eta} \in W_H(K)} \gamma^{j\eta} \cdot x = x + \gamma^\eta x + \cdots + \gamma^{A\eta}$$

where $A = p^{k-i} - 1$, and hence

$$\prod_{\gamma^t \in W_G(H)} \gamma^t \cdot \left(\sum_{\gamma^{j\eta} \in W_H(K)} \gamma^{j\eta} \cdot x \right) = \tag{1.4.4}$$
$$(x + \cdots + \gamma^{A\eta} x)(\gamma x + \cdots + \gamma^{A\eta+1}) \cdots (\gamma^{\eta-1} x + \cdots + \gamma^{(A+1)\eta-1} x).$$

Expanding Equation 1.4.4 results in a $(p^{k-1})^{p^{n-k}}$-many term polynomial in x and its $W_G(K)$-conjugates. Each monomial is a product of $|G/H|$-many variables, and if $\gamma^s x$ appears in a monomial then it appears no more than once in that term. In other words, if a monomial contains the element $\gamma^s x^m$ then $m = 1$. Moreover, this polynomial is universally determined by the group G and is independent of both the given Mackey functor and the given element x.

If we examine Equation 1.4.4 a bit further we will be able to obtain even more details in the description of $N_H^G(tr_K^H(x))$. To begin, the stabilizer subgroup of every term in the expansion of Equation 1.4.4 is the subgroup K. If a term were to be stabilized by a subgroup K' such that $K < K' < G$ then this term would contain multiple elements $\gamma^s x$ from the same factor. Since this does not happen every monomial is stabilized only by K. Hence, the expansion will divide into sums of $W_G(K)$-oribts, and we can rewrite it as

$$\begin{aligned}
&\prod_{\gamma^t \in W_G(H)} \gamma^t \cdot \left(\sum_{\gamma^{j\eta} \in W_H(K)} \gamma^{j\eta} \cdot x \right) && (1.4.5)\\
&= \sum_{\gamma^t \in W_G(K)} \gamma^t \cdot \vec{x}_1 + \sum_{\gamma^t \in W_G(K)} \gamma^t \cdot \vec{x}_2 + \cdots + \sum_{\gamma^t \in W_G(K)} \gamma^t \cdot \vec{x}_{p(k-i)p^{n-k}-n+i} \\
&= \sum_{\gamma^t \in W_G(K)} \gamma^t \cdot (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_{p(k-i)p^{n-k}-n+i})
\end{aligned}$$

where $\vec{x}_i$ is a representative from the i^{th} $W_G(K)$-orbit. But, since summing over the $W_G(K)$-action correlates to taking the transfer tr_K^G, Equation 1.4.5 translates to the following formula.

$$N_H^G(tr_K^H(x)) \quad = \quad tr_K^G(\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_{p^{(k-i)p^{n-k}-n+i}})$$

Furthermore, the summation $\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_{p^{(k-i)p^{n-k}-n+i}}$ is a polynomial in x and its $W_G(K)$-conjugates. Thus, we will denote it by $f(x)$, and whilst we cannot find a closed form expression for $f(x)$ we emphasize that this polynomial is universally determined by the group G, and each $\vec{x}_i$ is a product $\prod \gamma^t \cdot x$ that contains no elements of the form $\gamma^t x^m$ for $m > 1$. Therefore, we can describe $N_H^G(tr_K^H(x))$ by Equation 1.4.6.

$$N_H^G(tr_K^H(x)) \quad = \quad tr_K^G(f(x)) \tag{1.4.6}$$

Example 1.4.3. If the group G is small enough we can use Equation 1.4.6 to find an explicit formula for $N_H^G(tr_K^H(x))$. For example, for any C_8-Tambara functor we know via Equation 1.4.6 that $N_{C_4}^{C_8}(tr_e^{C_4}(x)) = tr_e^{C_8}(f(x))$ where $f(x)$ is some polynomial in x and its C_8-conjugates. However, letting γ be the generator of C_8, we can determine a closed form expression for $N_{C_4}^{C_8}(tr_e^{C_4}(x))$ by computing

$$(x + \gamma^2 x + \gamma^4 x + \gamma^6 x)(\gamma x + \gamma^3 x + \gamma^5 x + \gamma^7 x).$$

Indeed,

$$
\begin{aligned}
&(x+\gamma^2x+\gamma^4x+\gamma^6x)(\gamma x+\gamma^3x+\gamma^5x+\gamma^7x) = \\
&\qquad x\gamma x + x\gamma^3x + x\gamma^5x + x\gamma^7x + \gamma^2x\gamma^x + \gamma^2x\gamma^3x + \gamma^2x\gamma^5x + \gamma^2x\gamma^7x \\
&\quad + \quad \gamma^4x\gamma x + \gamma^4x\gamma^3x + \gamma^4x\gamma^5x + \gamma^4x\gamma^7x + \gamma^6x\gamma x + \gamma^6x\gamma^3x + \gamma^6x\gamma^5x + \gamma^6x\gamma^7x \\
&\quad = \quad \sum_{\gamma^t\in C_8} \gamma^t\cdot(x\gamma x + x\gamma^3x).
\end{aligned}
$$

It follows that the polynomial $f(x) = x\gamma x + x\gamma^3x$, and in any C_8-Tambara functor an explicit formula for $N_{C_4}^{C_8}(tr_e^{C_4}(x))$ is given by $N_{C_4}^{C_8}(tr_e^{C_4}(x)) = tr_e^{C_8}(x\gamma x + x\gamma^3x)$.

Property 1.4.4. If $H = C_{p^k}$, $K' = C_{p^i}$, $H < K' < G$ and m is an element in $\underline{M}(G/H)$ then by functoriality we can conclude that $N_H^G(m) = N_{K'}^G N_H^{K'}(m)$. Moreover,

$$
\begin{aligned}
\prod_{\gamma^s\in W_G(H)} \gamma^s\cdot m &= \prod_{s=0}^{p^{n-k}-1} \gamma^s\cdot m \\
&= \prod_{s=0}^{p^{n-i}-1}\left(\prod_{j=0}^{p^{n-i-k}-1} \gamma^{s+jp^{n-i}}\cdot m\right) \\
&= \prod_{\gamma^s\in W_G(K)} \gamma^s\cdot\left(\prod_{\gamma^t\in W_K(H)} \gamma^t\cdot m\right).
\end{aligned}
$$

Therefore, Equation 1.4.2 for $N_H^G(a+b)$ must agree with the formula for $N_K^G N_H^{K'}(a+b)$, and Equation 1.4.6 for $N_H^G(tr_{H'}^H(x))$ must agree with the formula for $N_{K'}^G N_H^{K'}(tr_{H'}^H(x))$.

1.4.3 A Definition for C_{p^n}-Tambara Functors

The above discussion provides a more pragmatic definition of a C_{p^n}-Tambara functor.

Definition 1.4.2. Let G be C_{p^n}. A commutative G-Green functor $\underline{S}$ is a G-*Tambara functor* (or just Tambara functor when the group is clear) if it has the following extra structure.

1. For all subgroups H of G, if K is a subgroup of H then, in addition to the restriction map res^H_K and transfer map tr^H_K, $\underline{S}$ supports a third map $N^H_K : \underline{S}(G/K) \to \underline{S}(G/H)$, called the *norm* map. The norm map is multiplicative but not additive.

2. Like the transfer map tr^H_K, the norm map $N^H_K = N^H_{K'} N^{K'}_K$ whenever $K < K' < H$ and $N^H_K(\gamma \cdot x) = N^H_K(x)$ for all x in $\underline{S}(G/K)$ and γ in $W_H(K)$.

3. Given x in $\underline{S}(G/K)$ and γ in $W_H(K)$, $res^H_K N^H_K(x) = \prod_{\gamma \in W_H(K)} \gamma \cdot x$ for all $K < H \leq G$.

4. If $K < H$ then for all a and b in $\underline{S}(G/K)$
$$N^H_K(a+b) = N^H_K(a) + N^H_K(b) + tr^H_K(g_K(a,b)) + \sum_{K<K'<H} tr^H_{K'}\left(\sum_j N^{K'}_K((\vec{ab})^{K'}_j)\right)$$
as described in Equation 1.4.2 of Remark 1.4.1. In particular, this equation is universally determined by the group G. It is independent of the given Tambara functor $\underline{S}$.

5. If $K < K' < H$ then for all x in $\underline{S}(G/K')$
$$N^H_K(tr^K_{K'}(x)) = tr^H_{K'}(f(x))$$

where $f(x)$ is the polynomial of Equation 1.4.6 of Remark 1.4.2. This polynomial is universally determined by the group G and does not depend on the given Tambara functor $\underline{S}$.

We will describe a Tambara functor diagrammatically, and as we did for Mackey functors, we will omit the Weyl action from Tambara functor diagrams. For example, a C_8-Tambara functor is shown in Figure 1.3.

$$
\begin{tikzcd}[row sep=large]
\underline{M}(C_8/C_8) \arrow[d, bend right=60, "res^{C_8}_{C_4}"'] \\
\underline{M}(C_8/C_4) \arrow[u, "N^{C_8}_{C_4}"'] \arrow[u, bend right=60, "tr^{C_8}_{C_4}"'] \arrow[d, bend right=60, "res^{C_4}_{C_2}"'] \\
\underline{M}(C_8/C_2) \arrow[u, "N^{C_4}_{C_2}"'] \arrow[u, bend right=60, "tr^{C_4}_{C_2}"'] \arrow[d, bend right=60, "res^{C_2}_{e}"'] \\
\underline{M}(C_8/e) \arrow[u, "N^{C_2}_{e}"'] \arrow[u, bend right=60, "tr^{C_2}_{e}"']
\end{tikzcd}
$$

Figure 1.3: A C_8-Tambara Functor

Furthermore, we can consider the category of G-Tambara functors $Tamb_G$, and a morphism in $Tamb_G$ is a morphism of commutative Green functors that also commutes with the norm maps. More specifically, a morphism $\phi : \underline{S} \to \underline{S}'$ of Tambara functors consists of a collection of Weyl equivariant ring homomorphisms $\phi_H : \underline{S}(G/H) \to \underline{S}'(G/H)$ for all subgroups H of G such that whenever K is a

subgroup of H $res^H_K\phi_H = \phi_K res^H_K$, $tr^H_K\phi_K = \phi_H tr^H_K$, and $N^H_K\phi_K = \phi_H N^H_K$.

1.4.4 Examples

Example 1.4.5. *Fixed Point Tambara Functors.* A fixed point Green functor naturally inherits the structure of a Tambara functor. If D is a commutative ring with a G-action and $\underline{D}$ its fixed point Green functor then for all subgroups $K < H < G$ we can define the norm map $N^H_K : D^K \to D^H$ by

$$N^H_K(x) = \prod_{\gamma \in W_H(K)} \gamma \cdot x.$$

For example, the constant Mackey functor extends to a Tambara functor, and we define its norm maps $N^H_K : \mathbb{Z} \to \mathbb{Z}$ by $x \mapsto x^{|H/K|}$ for all subgroups H and K in G. The C_{p^2}-Constant Tambara functor is given below.

$$\begin{tikzcd}
\underline{\mathbb{Z}}(C_{p^2}/C_{p^2}) = \mathbb{Z} \arrow[d, bend right=60, "id"'] \\
\underline{\mathbb{Z}}(C_{p^2}/C_p) = \mathbb{Z} \arrow[u, "N_{C_p}^{C_{p^2}}", "x^p"'] \arrow[u, bend right=60, "(\times p)"'] \arrow[d, bend right=60, "id"'] \\
\underline{\mathbb{Z}}(C_{p^2}/e) = \mathbb{Z} \arrow[u, "N_e^{C_p}", "x^p"'] \arrow[u, bend right=60, "(\times p)"']
\end{tikzcd}$$

Example 1.4.6. *The Burnside Tambara Functor.* The Burnside Green functor extends to a Tambara functor as well. If X is a K-set and K is a subgroup of H then the set

$$\{K\text{-equivariant maps } \phi : H \to X\}$$

is an H-set, and the H-action on a map ϕ in this set is given by $h \cdot \phi(h') = \phi(hh')$ for all h in H. Thus, we can define the norm maps $N_K^H : \underline{A}(G/K) \to \underline{A}(G/H)$ of the Burnside Tambara functor by

$$N_K^H([X]) = [\{K\text{-equivariant maps } \phi : H \to X\}].$$

We can provide an explicit formula for the norm $N_e^{C_p} : \underline{A}(C_p/e) \to \underline{A}(C_p/C_p)$ of the C_p-Burnside Tambara functor. From Example 1.3.2 we have $\underline{A}(C_p/e) = \mathbb{Z}$ and $\underline{A}(C_p/C_p) = \mathbb{Z}[t]/_{t^2=pt}$. Let M be a set of m-many elements, and let m denote the isomorphism class of M in $\underline{A}(C_p/e)$. Then

$$N_e^{C_p}(m) = [\{\text{maps } C_p \to M\}],$$

and there are m^p-many such maps, m-many of which are constant. Thus, each constant map is in its own orbit, and the remaining $(m^p - m)$-many maps split into $\frac{m^p-m}{p}$ copies of $[C_p/e]$. Therefore, $N_e^{C_p}(m) = m + \frac{m^p-m}{p}t$, and we show the C_p-Burnside Tambara functor diagram below.

$$\begin{tikzcd}
1 \arrow[d, mapsto] & t \arrow[d, mapsto] & \mathbb{Z}[t]/_{t^2=pt} \arrow[d, bend right=50, "res_e^{C_p}"'] & t \\
1 & p & \mathbb{Z} \arrow[u, "N_e^{C_p}"] \arrow[u, bend right=50, "tr_e^{C_p}"'] & 1 \arrow[u, mapsto]
\end{tikzcd}$$

Example 1.4.7. *Not All Green Functors are Tambara Functors.* If $G = C_{p^n}$ then there is an augmentation map of G-Green functors $\phi : \underline{A} \to \underline{\mathbb{Z}}$ where each ϕ_H :

$\underline{A}(G/H) \to \mathbb{Z}$ sends a finite H-set to its cardinality. We define the *augmentation ideal* $\underline{I}$ to be the kernel of this map, and the Mackey functor diagram for $\underline{I}$ is shown below.

$$\begin{tikzcd} \underline{I}(C_p/C_p) = \mathbb{Z} \arrow[d, bend right, "res_e^{C_p}"'] \\ \underline{I}(C_p/e) = 0 \arrow[u, bend right, "tr_e^{C_p}"'] \end{tikzcd}$$

The augmentation ideal is a Green functor that does not extend to a Tambara functor because $\underline{I}(C_p/e) = 0$. Since the norm map is multiplicative we require $N_e^{C_p}(1) = 1$, but $1 = 0$ in $\underline{I}(C_p/e)$. Thus, if $\underline{I}$ supported a norm map then it would follow that $1 = 0$ in $\underline{I}(C_p/C_p)$. Indeed this is not the case.

Example 1.4.8. *Using an Adjunction to Build Tambara Functors.* Let $Ring_G$ be the category of commutative rings with a G-action. There is a forgetful functor $Tamb_G \to Ring_G$ given by $\underline{S} \mapsto \underline{S}(G/e)$, and this functor has a left adjoint

$$F_T : Ring_G \to Tamb_G.$$

Thus, we can use this adjunction to build Tambara functors from rings in $Ring_G$.

In particular, if R is a nice ring in $Ring_{C_p}$ then we can build a Tambara functor diagram for the C_p-Tambara functor $F_T(R)$. We define $F_T(R)(C_p/e)$ to be R, and we can determine $F_T(R)(C_p/C_p)$ as follows. First, the ring $F_T(R)(C_p/C_p)$ must consist of a copy of the Burnside Tambara functor $\underline{A}$ so that we can define $tr_e^{C_p}(1)$ to be t. Then for every additive generator r_i in R we add polynomial generators u_i and n_i to

$F_T(R)(C_p/C_p)$ and define the transfer and norm maps by

$$tr_e^{C_p}(r_i) = u_i \text{ and } N_e^{C_p}(r_i) = n_i.$$

We further force the Frobenius reciprocity relations and use the Weyl action to define the restriction map.

For example, we will construct the C_2-Tambara functor $F_T(\mathbb{Z}/2)$ where we consider $\mathbb{Z}/2$ as a ring with trivial C_2-action. We define $F_T(\mathbb{Z}/2)(C_2/C_2)$ to be

$$\underline{A}(C_2/C_2)/_{tr_e^{C_2}(2)=0, N_e^{C_2}=0}.$$

Since

$$\begin{aligned} \underline{A}(C_2/C_2) &= \mathbb{Z}[t]/_{t^2=2t}, \\ tr_e^{C_2}(2) &= 2t, \text{ and} \\ N_e^{C_2}(2) &= 2+t \end{aligned}$$

it follows that $F_T(\mathbb{Z}/2)(C_2/C_2) = \mathbb{Z}[t]/_{t^2=2t, 2t=0, 2+t=0}$, but we can simplify this further. The relation $2+t=0$ implies that $t=-2$ and $2t=0$ implies that $4=0$. Therefore, the ring $F_T(\mathbb{Z}/2)(C_2/C_2)$ is isomorphic to $\mathbb{Z}/4$, and $tr_e^{C_2}(1) = 2$. We give the C_2-Tambara functor diagram for $F_T(\mathbb{Z}/2)$ below.

$$\begin{tikzcd}
1 \arrow[d, mapsto] & \mathbb{Z}/4 \arrow[d, bend right=50, "res_e^{C_2}"'] & 2 \\
1 & \mathbb{Z}/2 \arrow[u, "N_e^{C_2}"] \arrow[u, bend right=50, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto]
\end{tikzcd}$$

Next, consider the ring $\mathbb{Z}[x]/x^2$ with trivial C_2-action. We will build the C_2-Tambara functor $F_T(\mathbb{Z}[x]/x^2)$. To define $F_T(\mathbb{Z}[x]/x^2)(C_2/C_2)$ we must adjoin $tr_e^{C_2}(x)$ and $N_e^{C_2}(x)$ to $\underline{A}(C_2/C_2)$ and still maintain Frobenius reciprocity and all relations between the restriction, transfer and norm maps given in Definition 1.4.2. Thus, if we let $tr_e^{C_2}(x) = u$ and $N_e^{C_2}(x) = n$ then since the Weyl action is trivial, we require

$$res_e^{C_2}(u) = res_e^{C_2} tr_e^{C_2}(x) = \sum_{\gamma \in C_2} \gamma \cdot x = 2x$$

and

$$res_e^{C_2}(n) = res_e^{C_2} N_e^{C_2}(x) = \prod_{\gamma \in C_2} \gamma \cdot x = x^2 = 0.$$

Moreover, Frobenius reciprocity induces the relations shown below.

$$
\begin{array}{ccccccccc}
tu & = & tr(1)u & = & tr(1 \cdot res(u)) & = & tr(2x) & = & 2y \\
tn & = & tr(1)n & = & tr(1 \cdot res(n)) & = & tr(x^2) & = & 0 \\
nu & = & ntr(x) & = & tr(res(n)x) & = & tr(x^3) & = & 0 \\
u^2 & = & tr(x)u & = & tr(xres(u)) & = & tr(2x^2) & = & 0
\end{array}
$$

Therefore,

$$F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) = \mathbb{Z}[t,u,n]/_{(t^2=2t, tu=2u, tn=nu=u^2=n^2=0)},$$

and we give the C_2-Tambara functor diagram for $F_T(\mathbb{Z}[x]/x^2)$.

$$
\begin{tikzcd}
1 \arrow[d, mapsto] & t \arrow[d, mapsto] & u \arrow[d, mapsto] & n \arrow[d, mapsto] & F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[d, bend right=60, "res_e^{C_2}"'] & t & u \\
1 & 2 & 2x & 0 & \mathbb{Z}[x]/x^2 \arrow[u, "N_e^{C_2}"] \arrow[u, bend right=60, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto] & x \arrow[u, mapsto]
\end{tikzcd}
$$

1.4.5 The Box Product of Tambara Functors

In ([12], Proposition 9.1) Neil Strickland states that if $\underline{S}$ and $\underline{S}'$ are Tambara functors then the Mackey functor $\underline{S} \square \underline{S}'$ inherits a unique Tambara functor structure. We define the norm maps $N_K^H : (\underline{S} \square \underline{S}')(G/K) \to (\underline{S} \square \underline{S}')(G/H)$ by letting $N_K^H(a \otimes b) = N_K^H(a) \otimes N_K^H(b)$ if $a \otimes b$ is in $\underline{S}(G/K) \otimes \underline{S}'(G/K)$. We then use Properties 4 and 5 of Definition 1.4.2 to extend this definition to all other elements in $(\underline{S} \square \underline{S}')(G/K)$.

Hence, if $\underline{S}$ and $\underline{S}'$ are C_{p^n}-Tambara functors then $(\underline{S} \square \underline{S}')(G/H)$ still equals

$$\left[\underline{S}(G/H) \otimes \underline{S}'(G/H) \oplus Im(tr_K^H)\right] /_{FR}.$$

Given $a \otimes b$ in $\underline{S}(G/H) \otimes \underline{S}'(G/H)$ and $tr_K^H(x)$ in $Im(tr_K^H)$ we define the multiplication $(a \otimes b)tr_K^H(x)$ using Frobenius reciprocity:

$$(a \otimes b)tr_K^H(x) = tr_K^H(res_K^H(a \otimes b)x).$$

Moreover, Remarks 1.2.2 and 1.2.3 extend to the box product of Tambara functors. In particular, a map $\underline{S} \square \underline{S}' \to \underline{R}$ of Tambara functors determines and is determined by a collection of ring homomorphisms

$$\theta_H : \underline{S}(G/H) \otimes \underline{S}'(G/H) \to \underline{R}(G/H)$$

for all subgroups H of G such that whenever K is a subgroup of H the diagrams of Remark 1.2.3 commute *and* the diagram below commutes.

$$\begin{tikzcd} \underline{S}(G/H) \otimes \underline{S}'(G/H) \arrow[r, "\theta_H"] & \underline{R}(G/H) \\ \underline{S}(G/K) \otimes \underline{S}'(G/K) \arrow[u, "N_K^H \otimes N_K^H"] \arrow[r, "\theta_K"] & \underline{R}(G/K) \arrow[u, "N_K^H"'] \end{tikzcd}$$

Chapter 2

An Equivariant Symmetric Monoidal Structure on the Category of Mackey Functors

We now develop a new equivariant symmetric monoidal structure on the category of Mackey functors under which Tambara functors are the equivariant commutative monoids. The main advantage of this new structure is that it is concrete. We will be able to build a Mackey functor diagram like Figure 1.1 that describes this G-symmetric monoidal structure just like the diagram in Example 1.2.1 describes the box product construction.

2.1 G-Symmetric Monoidal Structures

In this section we provide the definition G-symmetric monoidal and G-commutative monoid given by Hill and Hopkins in [4], but first we discuss a useful property of symmetric monoidal categories. Let $\mathscr{S}et^{Fin,Iso}$ be the category whose objects are finite sets (with no group action) and whose morphisms consist only of isomorphisms

of sets, and let $(\mathscr{C}, \boxtimes, e)$ be a symmetric monoidal category. Because $\mathscr{C}$ is symmetric monoidal there is a functor $(-) \otimes (-) : \mathscr{S}et^{Fin,Iso} \times \mathscr{C} \to \mathscr{C}$ given by

$$X \otimes \mathscr{C} = \underbrace{C \boxtimes \cdots \boxtimes C}_{|X| \text{ times}} = C^{\boxtimes |X|}$$

called the *canonical exponentiation map* [4]. It is symmetric monoidal in both factors. Additionally, the object $\emptyset \otimes C = e$, and if $*$ is the single point set then $* \otimes C = C$ for all C in $\mathscr{C}$.

We can use this functor to describe the commutative monoids in $\mathscr{C}$. If we fix an object C in $\mathscr{C}$ then we can define a functor $(-) \otimes C : \mathscr{S}et^{Fin,Iso} \to \mathscr{C}$ by $X \mapsto X \otimes C$. We can define a commutative monoid to be an object C in $\mathscr{C}$ along with an extension of the functor $(-) \otimes C$ to $\mathscr{S}et^{Fin}$, the category of finite sets.

$$\begin{tikzcd}
\mathscr{S}et^{Fin,Iso} \arrow[r, "(-)\otimes C"] \arrow[d] & \mathscr{C} \\
\mathscr{S}et^{Fin} \arrow[ur, dashed] &
\end{tikzcd}$$

If such an extension exists then morphisms $X \to Y$ in $\mathscr{S}et^{Fin}$ induce morphisms $X \otimes C \to Y \otimes C$, and we recover the standard definition of a commutative monoid by examining the morphisms in $\mathscr{C}$ induced from $* \amalg * \to *$, $\emptyset \to *$, and $* \amalg * \to * \amalg *$ [4].

We develop the notions of G-symmetric monoidal and G-commutative monoid by extending this discussion to $\mathscr{S}et_G^{Fin,Iso}$, the category of finite G-sets with isomorphisms.

Definition 2.1.1. Let $(\mathscr{C}, \boxtimes, e)$ be a symmetric monoidal category. A *G-symmetric monoidal structure* on $\mathscr{C}$ consists of a functor

$$(-) \otimes (-) : \mathscr{S}et_G^{Fin,Iso} \times \mathscr{C} \to \mathscr{C}$$

such that

1. $(X \amalg Y) \otimes C = (X \otimes C) \boxtimes (Y \otimes C)$ and $X \otimes (C \boxtimes D) = (X \otimes C) \boxtimes (X \otimes D)$,

2. when restricted to $\mathscr{S}et^{Fin,Iso}$ this functor is the canonical exponentiation map, and

3. $X \otimes (Y \otimes C)$ is naturally isomorphic to $(X \times Y) \otimes C$.

As with the non-equivariant case, every object C in $\mathscr{C}$ defines a functor

$$(-) \otimes C : \mathscr{S}et_G^{Fin,Iso} \to \mathscr{C},$$

and so the definition of a G-commutative monoid is analogous to the definition of a commutative monoid presented above.

Definition 2.1.2. A *G-commutative monoid* is an object C in $\mathscr{C}$ together with an extension

$$\begin{tikzcd} \mathscr{S}et_G^{Fin,Iso} \arrow[r, "(-)\otimes C"] \arrow[d] & \mathscr{C} \\ \mathscr{S}et_G^{Fin} \arrow[ur, dashed] & \end{tikzcd}$$

Example 2.1.1. Let $\mathcal{S}p^G$ be the category of G-spectra, and for every subgroup H of G let $i_H^* : \mathcal{S}p^G \to \mathcal{S}p^H$ be the restriction functor that sends a G-spectrum to its

underlying H-spectrum. The smash product makes $\mathcal{S}p^G$ into a symmetric monoidal category.

We can endow $\mathcal{S}p^G$ with a G-symmetric monoidal structure using the Hill-Hopkins-Ravenel norm $N_H^G : \mathcal{S}p^H \to \mathcal{S}p^G$. We define the functor

$$(-) \otimes (-) : \mathscr{S}et_G^{Fin,Iso} \times \mathcal{S}p^G \to \mathcal{S}p^G$$

by

$$\begin{aligned} G/H \otimes X &= N_H^G i_H^* X \text{ for all orbits } G/H \text{, and} \\ (Y \amalg Z) \otimes X &= (Y \otimes X) \wedge (Z \otimes X) \text{ for all finite } G\text{-sets } Y \text{ and } Z. \end{aligned}$$

Then given a G-spectrum X the functor $(-) \otimes X : \mathscr{S}et_G^{Fin,Iso} \to \mathcal{S}p^G$ extends to a functor $\mathscr{S}et_G^{Fin} \to \mathcal{S}p^G$ if and only if X has the structure of a commutative G-ring spectrum. Therefore, under this G-symmetric monoidal structure the commutative G-ring spectra are the G-commutative monoids [5].

We can use the above G-symmetric monoidal structure on the category of G-spectra to build a G-symmetric monoidal structure on the category of G-Mackey functors. We define a functor $(-) \otimes (-) : \mathscr{S}et_G^{Fin,Iso} \times \mathcal{M}ack_G \to \mathcal{M}ack_G$ by

$$(X, \underline{M}) \mapsto \underline{\pi_0}(X \otimes H\underline{M})$$

where $H\underline{M}$ is the Eilenberg-MacLane spectrum of $\underline{M}$ [4].

However, this G-symmetric monoidal structure is not ideal. In addition to being difficult to unpack, we have been unable to show that Tambara functors are the G-commutative monoids under this structure. In 2004, Morten Brun showed that if R

is a commutative G-ring spectrum then $\underline{\pi_0}(R)$ is a Tambara functor [1], and hence, we know that all G-commutative monoids are Tambara functors. However, it is not clear that all Tambara functors are G-commutative monoids.

Therefore, we strive to build a new G-symmetric monoidal structure on $\mathcal{M}ack_G$ such that

- this new structure is concrete and does not involve the passage to G-spectra, and
- under this structure a Mackey functor $\underline{M}$ is a G-commutative monoid *if and only if* it has the structure of a Tambara functor.

We accomplish this endeavor for the category of C_{p^n}-Mackey functors by creating symmetric monoidal *norm functors* $N_H^{C_{p^n}} : \mathcal{M}ack_H \to \mathcal{M}ack_{C_{p^n}}$ for all subgroups H of C_{p^n} such that

- given an H-Mackey functor $\underline{M}$ we can write down a Mackey functor diagram for $N_H^{C_{p^n}} \underline{M}$, and
- $N_H^{C_{p^n}} \underline{M}$ is the universal home for internal norms of Tambara functors.

If $G = C_{p^n}$ then we will be able to define the functor

$$(-) \otimes (-) : \mathscr{S}et_G^{Fin,Iso} \times \mathcal{M}ack_G \to \mathcal{M}ack_G$$

by

- $\emptyset \otimes \underline{M} := \underline{A}$, the Burnside Mackey functor,

- $G/H \otimes \underline{M} := N_H^G i_H^* \underline{M}$, for all orbits G/H and
- $(X \amalg Y) \otimes \underline{M} := (X \otimes \underline{M}) \square (Y \otimes \underline{M})$ for all finite G-sets X and Y.

The functor $i_H^* : \mathcal{M}ack_G \to \mathcal{M}ack_H$ is the restriction functor that brings a G-Mackey functor to its underlying H-Mackey functor. For example, if $\underline{M}$ is the C_8-Mackey functor in Figure 1.1 then the Mackey functor diagram for $i_{C_2}^*\underline{M}$ is shown below.

$$\begin{tikzcd}
(i_{C_2}^*\underline{M})(C_2/C_2) = \underline{M}(C_8/C_2) \arrow[d, bend right, "res_e^{C_2}"'] \\
(i_{C_2}^*\underline{M})(C_2/e) = \underline{M}(C_8/e) \arrow[u, bend right, "tr_e^{C_2}"']
\end{tikzcd}$$

In all subsequent sections let G be C_{p^n}.

2.2 Constructing the Norm Functors N_H^G

2.2.1 Part 1: H is Maximal in G

We will first build the norm functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ for $H = C_{p^{n-1}}$, the maximal subgroup of G. We need to construct this functor such that $i_K^*(X \otimes \underline{M})$ is isomorphic to $i_K^*X \otimes i_K^*\underline{M}$ for all subgroups K of G, finite G-sets X and Mackey

functors $\underline{M}$. In particular, we need $i_H^*(G/H \otimes \underline{M})$ to equal $i_H^*(G/H) \otimes i_H^*\underline{M}$ and

$$\begin{aligned} i_H^*(G/H) \otimes i_H^*\underline{M} &= \underbrace{(H/H \amalg \cdots \amalg H/H)}_{|G/H| \text{ times}} \otimes i_H^*\underline{M} \\ &\cong (H/H \otimes i_H^*\underline{M})^{\square |G/H|} \\ &\cong (i_H^*\underline{M})^{\square |G/H|}. \end{aligned}$$

Thus, letting $\underline{M}$ be an H-Mackey functor, the Mackey functor $i_H^* N_H^G \underline{M}$ must equal $\underline{M}^{\square |G/H|}$, and so if K is a subgroup of H we define $(N_H^G\underline{M})(G/K)$ to be $\underline{M}^{\square |G/H|}(H/K)$ where the Weyl group $W_G(K)$ acts by permuting the box product factors. More specifically, if γ is the generator of C_{p^n} and $m_e \otimes m_\gamma \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-1}}$ is a simple tensor in $\underline{M}^{\square |G/H|}(H/K)$ then $W_G(K)$ acts by

$$\gamma \cdot (m_e \otimes m_\gamma \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-1}}) = (\gamma^p \cdot m_{\gamma^{p-1}}) \otimes m_e \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-2}}$$

where we regard γ^p as the generator of $W_H(K)$.

Remark 2.2.1. We will often consider a group M as a trivial group-Mackey functor, and hence we can construct the G-Mackey functor $N_e^G M$. In this case $(N_e^G M)(G/e) = M^{\otimes |G|}$. The group G still acts on $M^{\otimes |G|}$ by permuting the tensor factors.

Remark 2.2.2. For any subgroup K of H we will want to write a simple tensor

$$m_e \otimes m_\gamma \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-1}}$$

in $\underline{M}^{\square |G/H|}(H/K)$ as a product over the Weyl action. Of course, the group $\underline{M}(H/K)$ does not necessarily come equipped with a multiplication. Thus, we must formally

add an element 1 to $\underline{M}(H/K)$ such that $\gamma^t \cdot 1 = 1$ for all γ^t in G, and we define a multiplication on 1 by $1m = m1 = 1$ for all m in $\underline{M}(H/K)$. If neither m_1 nor m_2 are the element 1 then there is no multiplication $m_1 m_2$. Then we can express the element

$$m_e \otimes m_\gamma \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-1}}$$

as

$$(m_e \otimes 1^{\otimes p-1})\gamma(m_\gamma \otimes 1^{\otimes p-1})\gamma^2(m_{\gamma^2} \otimes 1^{\otimes p-1}) \cdots \gamma^{p-1}(m_{\gamma^{p-1}} \otimes 1^{\otimes p-1}).$$

Furthermore, consider the polynomials $g_H(a,b)$ of Equation 1.4.3 in Section 1.4.1 and $f(x)$ of Equation 1.4.6 in Section 1.4.2. The polynomial $g_H(a,b)$ is a polynomial in a, b, and their $W_G(H)$-conjugates. Since every monomial of $g_H(a,b)$ contains only elements of the form $\gamma^s a$ and $\gamma^s b$ we can make sense of evaluating this polynomial at the elements $a \otimes 1^{\otimes p-1}$ and $b \otimes 1^{\otimes p-1}$ and their $W_G(H)$-conjugates in $\underline{M}^{\Box|G/H|}(H/H)$.

For example, in Example 1.4.1 we computed

$$N_e^{C_2}(a+b) = N_e^{C_2}(a) + N_e^{C_2}(b) + tr_e^{C_2}(g_e(a,b))$$

where $g_e(a,b) = a\gamma b$. If M is a group then $(N_e^{C_2}M)(C_2/e) = M \otimes M$, and hence below we evaluate $g_e(a,b)$ at $a \otimes 1$ and $b \otimes 1$ in $M \otimes M$.

$$\begin{aligned} g_e(a \otimes 1, b \otimes 1) &= (a \otimes 1)\gamma(b \otimes 1) \\ &= (a \otimes 1)(1 \otimes b) \\ &= a \otimes b \end{aligned}$$

Similarly, the polynomial $f(x)$ is a polynomial in x and its $W_G(K)$-conjugates, and if $\gamma^s x^m$ is part of a monomial of $f(x)$ then $m = 1$. Therefore, we can evaluate

$f(x)$ at $x \otimes 1^{\otimes p-1}$ and its $W_G(K)$-conjugates in $\underline{M}^{\square|G/H|}(H/K)$. Indeed, in Example 1.4.3 we showed that $N_{C_4}^{C_8}(tr_e^{C_4}(x)) = tr_e^{C_8}(f(x))$, where $f(x) = x\gamma x + x\gamma^3 x$. Thus, evaluating $f(x)$ at $x \otimes 1$ in $\underline{M}^{\square|C_8/C_4|}(C_4/e)$ results in

$$\begin{aligned} f(x \otimes 1) &= (x \otimes 1)\gamma(x \otimes 1) + (x \otimes 1)\gamma^3(x \otimes 1) \\ &= (x \otimes 1)(1 \otimes x) + (x \otimes 1)(1 \otimes \gamma^2 x) \\ &= x \otimes x + x \otimes \gamma^2 x. \end{aligned}$$

We now provide a complete definition of the norm functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ for H maximal in G. We will prove that our construction of N_H^G is, in fact, a functor in Theorem 2.2.3.

Definition 2.2.1. Define the norm functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ as follows. Given an H-Mackey functor $\underline{M}$, for all subgroups K of H define

$$(N_H^G \underline{M})(G/K) := \underline{M}^{\square|G/H|}(H/K).$$

The Weyl group $W_G(K)$ acts on a simple tensor by

$$\gamma \cdot (m_e \otimes m_\gamma \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-1}}) = (\gamma^p \cdot m_{\gamma^{p-1}}) \otimes m_e \otimes m_{\gamma^2} \otimes \cdots \otimes m_{\gamma^{p-2}}$$

where γ^p is the generator of $W_H(K)$. If $K' < K \leq H$ then the restriction map $res_{K'}^K$ and the transfer map $tr_{K'}^K$ are defined as in Definition 1.2.1. Define

$$(N_H^G \underline{M})(G/G) := \left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \underline{M}^{\square|G/H|}(H/H)/_{W_G(H)}\right)/_{TR}.$$

1. For a in $\underline{M}(H/H)$ let $N(a)$ denote the corresponding generator of the free summand $\mathbb{Z}\{\underline{M}(H/H)\}$.

2. The transfer map tr_H^G is the quotient map

$$\underline{M}^{\Box|G/H|}(H/H) \to \underline{M}^{\Box|G/H|}(H/H)/_{W_G(H)}$$

onto the second summand. We will refer to $\underline{M}^{\Box|G/H|}(H/H)/_{W_G(H)}$ as $Im(tr_H^G)$ and an element in $\underline{M}^{\Box|G/H|}(H/H)/_{W_G(H)}$ as $tr_H^G(x)$.

3. We define the restriction map res_H^G by $res_H^G(tr_H^G(x)) = \sum_{\gamma \in W_G(H)} \gamma \cdot x$ and $res_H^G(N(a)) = a^{\otimes|G/H|}$.

4. The submodule TR is called the *Tambara reciprocity* submodule and is generated by

$$N(a+b) - N(a) - N(b) - tr_H^G(g(a,b))$$

and

$$N(tr_K^H(x)) - tr_H^G\left(tr_K^H(F(x))\right)$$

for all a and b in $\underline{M}(H/H)$ and all x in every $\underline{M}(H/K)$. The polynomial $g(a,b) = g_H(a \otimes 1^{\otimes p-1}, b \otimes 1^{\otimes p-1})$ where $g_H(a,b)$ is the polynomial of Equation 1.4.2 and Remark 2.2.2. The polynomial $F(x) = f(x \otimes 1^{\otimes p-1})$ where $f(x)$ is the polynomial given in Equation 1.4.6 and Remark 2.2.2. Both polynomials $g(a,b)$ and $F(x)$ are universally determined by the group G.

Definition 2.2.2. Define the map $N : \underline{M}(H/H) \to (N_H^G\underline{M})(G/G)$ by letting $N(a)$ be the corresponding generator in the free summand $\mathbb{Z}\{\underline{M}(H/H)\}$.

Definition 2.2.1 is reminiscent of Definition 1.2.1 of the box product $\underline{M} \,\square\, \underline{M}$. We need $(N_H^G\underline{M})(G/G)$ to be as free as possible such that the map N behaves like a norm map in a Tambara functor. In particular, if $\underline{L}$ is a G-Mackey functor and the functor $(-)\otimes\underline{L} : \mathscr{S}et_G^{Fin,Iso} \to \mathcal{M}ack_G$ extends to a functor $\mathscr{S}et_G^{Fin} \to \mathcal{M}ack_G$ then we want this map $N : i_H^*\underline{L}(H/H) \to (N_H^G i_H^*\underline{L})(G/G)$ to extend to an internal norm map $N_H^G : \underline{L}(G/H) \to \underline{L}(G/G)$ in $\underline{L}$. Thus, we wanted to let $(N_H^G\underline{M})(G/G)$ simply be $\mathbb{Z}\{\underline{M}(H/H)\}$. But, this definition was problematic because we were unable to define the transfer map $tr_H^G : (N_H^G\underline{M})(G/H) \to (N_H^G\underline{M})(G/G)$ in a compatible manner. So, instead, we needed to add in the image of the transfer as freely as possible and create Definition 2.2.1.

Moreover, since we want the map N to lead to internal norm maps we will think of the $\mathbb{Z}\{\underline{M}(H/H)\}$ summand of $(N_H^G\underline{M})(G/G)$ as the submodule of norms. We quotient out by the Tambara reciprocity submodule to force $(N_H^G\underline{M})(G/G)$ to maintain the relations that exist between the norm and transfer maps of a Tambara functor stated in Properties 4 and 5 of Definition 1.4.2. Similarly, we defined $res_H^G(N(a))$ to be $a^{\otimes|G/H|}$ so that it mirrors the restriction of a norm in a Tambara functor.

Example 2.2.3. Let $\underline{M}$ be a C_2-Mackey functor. Then $N_{C_2}^{C_4}\underline{M}$ is shown below.

$$\begin{tikzcd}
\Big(\mathbb{Z}\{\underline{M}(C_2/C_2)\} \oplus \overbrace{(\underline{M}\,\square\,\underline{M})(C_2/C_2)/_{W_{C_4}(C_2)}}^{Im(tr_{C_2}^{C_4})}\Big)/_{TR} \arrow[d, bend right, "res_{C_2}^{C_4}"'] \\
(\underline{M}\,\square\,\underline{M})(C_2/C_2) \arrow[u, bend right, "tr_{C_2}^{C_4}"'] \arrow[d, bend right, "res_e^{C_2}"'] \\
\underline{M}(C_2/e) \otimes \underline{M}(C_2/e) \arrow[u, bend right, "tr_e^{C_2}"']
\end{tikzcd}$$

The Tambara reciprocity submodule is generated by $N(a+b)-N(a)-N(b)-tr_{C_2}^{C_4}(a\otimes b)$ and $N(tr_e^{C_2}(x)) - tr_{C_2}^{C_4}tr_e^{C_2}(x\otimes x)$.

Example 2.2.4. Let $\underline{A}$ be the Burnside Mackey functor. Then $N_H^G\underline{A} = \underline{A}$. In particular, since $\mathbb{Z}$ is the Burnside Mackey functor of the trivial group it follows that $N_e^{C_2}\mathbb{Z}$ should be the C_2-Burnside Mackey functor. Indeed, $(N_e^{C_2}\mathbb{Z})(C_2/e) = \mathbb{Z}\otimes\mathbb{Z} = \mathbb{Z}$, and

$$(N_e^{C_2}\mathbb{Z})(C_2/C_2) = \left(\mathbb{Z}\{\dots, N(-1), N(0), N(1), \dots\} \oplus \mathbb{Z}\right)/_{TR}.$$

Then given any group M, the Tambara reciprocity submodule of $N_e^{C_2}M$ is generated by elements of the form $N(a+b)-N(a)-N(b)-tr(a\otimes b)$. Therefore, quotienting out by TR induces the following relations in $(N_e^{C_2}\mathbb{Z})(C_2/C_2)$.

$$N(0) = N(0) + N(0) + tr(0)$$

$$N(0) = N(a) + N(-a) + tr(-a^2) \text{ for all positive integers } a$$

$$N(c) = cN(1) + tr\left(\frac{c^2-c}{2}\right) \text{ for all integers } c \geq 2$$

These relations indicate that $N_e^{C_2}(C_2/C_2) = \mathbb{Z}\{N(1)\} \oplus \mathbb{Z}$ because $N(0) = 0$ and we can write all other generators as linear combinations of $N(1)$ and transfer terms. Further, $tr(1) = (0, 1)$, $res(1, 0) = res(N(1)) = 1 \otimes 1 = 1$, and $res(0, 1) = res(tr(1)) = \sum_{\gamma \in C_2} \gamma \cdot 1 = 2$. It follows that $N_e^{C_2}\mathbb{Z}$ is the C_2-Burnside Mackey functor.

Example 2.2.5. To stress the fact that this definition of N_H^G is concrete we will also build the Mackey functor diagram for $N_e^{C_2}(\mathbb{Z}/2)$. First, $N_e^{C_2}(\mathbb{Z}/2)(C_2/e) = \mathbb{Z}/2$, and

$$N_e^{C_2}(\mathbb{Z}/2)(C_2/C_2) = (\mathbb{Z}\{N(0), N(1)\} \oplus \mathbb{Z}/2) /_{TR}.$$

But, by Tambara reciprocity we have $N(0) = 0$ and

$$N(0) = N(1 + 1) = N(1) + N(1) + tr(1).$$

Thus, we can write $tr(1)$ in terms of $N(1)$. Finally, since $2N(1) + tr(1) = 0$ it follows that

$$2(2N(1) + tr(1)) = 4N(1) = 0,$$

which tells us that $N_e^{C_2}(\mathbb{Z}/2)(C_2/C_2)$ is isomorphic to $\mathbb{Z}/4$, and its generator is $N(1)$. The C_2-Mackey functor diagram for $N_e^{C_2}(\mathbb{Z}/2)$ is

$$\begin{tikzcd} 1 \arrow[d, mapsto] & \mathbb{Z}/4 \arrow[d, bend right=60, "res_e^{C_2}"'] & 2 \\ 1 & \mathbb{Z}/2 \arrow[u, bend right=60, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto] \end{tikzcd}$$

Now, we have defined a map $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ that is easy to construct for H maximal in G, but we need to prove that it is a symmetric monoidal functor.

Theorem 2.2.3. *The map $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ is a functor.*

Proof. Let $\underline{M}$ and $\underline{L}$ be H-Mackey functors. Given a morphism $\phi : \underline{M} \to \underline{L}$ of H-Mackey functors we need to define a morphism $\Phi : N_H^G\underline{M} \to N_H^G\underline{L}$ of G-Mackey functors. Thus, we must define a collection of maps $\Phi_K : (N_H^G\underline{M})(G/K) \to (N_H^G\underline{L})(G/K)$ for all subgroups K of G such that $res^K_{K'}\Phi_K = \Phi_{K'}res^K_{K'}$ and $tr^K_{K'}\Phi_{K'} = \Phi_K tr^K_{K'}$ whenever $K' < K$. If K is a subgroup of H then Φ_K is simply $\phi_K^{\square|G/H|}$, and then since H is maximal in G it remains only to define

$$\Phi_G : \left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus Im(tr_H^G)\right)/_{TR} \to \left(\mathbb{Z}\{\underline{L}(H/H)\} \oplus Im(tr_H^G)\right)/_{TR}.$$

In order for $tr_H^G\Phi_H$ to equal $\Phi_G tr_H^G$ we must define

$$\Phi_G(tr_H^G(x)) := tr_H^G(\phi_H^{\square|G/H|}(x)).$$

Then we define $\Phi_G(N(a))$ as follows.

$$\Phi_G(N(a)) := N(\phi_H(a)).$$

The map Φ_G will be well-defined if it preserves the Tambara reciprocity relations. More specifically, the following relations must hold for all a, b, and x.

$$\Phi_G(N(a+b)) = \tag{2.2.1}$$

$$\Phi_G(N(a)) + \Phi_G(N(b)) + \Phi_G(tr_H^G(g(a,b)))$$

$$\Phi_G(N(tr_K^H(x))) \;=\; \Phi_G\left[tr_H^G tr_K^H(F(x))\right] \tag{2.2.2}$$

To show that Equation 2.2.1 holds we have

$$\begin{aligned}
\Phi_G(N(a+b)) &= N(\phi_H(a+b)) \\
&= N(\phi_H(a)+\phi_H(b)) \\
&= N(\phi_H(a))+N(\phi_H(b))+tr_H^G(g(\phi_H(a),\phi_H(b)))
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_G(N(a))+\Phi_G(N(b))+\Phi_G(tr_H^G(g(a,b))) \\
&= N(\phi_H(a))+N(\phi_H(b))+tr_H^G(\phi_H^{\square|G/H|}(g(a,b))).
\end{aligned}$$

Since the polynomial $g(a,b)$ is in $\underline{M}^{\square|G/H|}(H/H)$ and is universally defined by the group G it follows that $\phi_H^{\square|G/H|}(g(a,b)) = g(\phi_H(a),\phi_H(b))$.

Similarly,

$$\begin{aligned}
\Phi_G(N(tr_K^H(x))) &= N(\phi_H(tr_K^H(x))) \\
&= N(tr_K^H(\phi_K(x))) \\
&= tr_H^G tr_K^H(F(\phi_K(x)))
\end{aligned}$$

and

$$\begin{aligned}
\Phi_G(tr_H^G tr_K^H(F(x))) &= tr_H^G tr_K^H(\Phi_K(F(x))) \\
&= tr_H^G tr_K^H(\phi_K^{\square|G/H|}(F(x))).
\end{aligned}$$

The polynomial $F(x)$ lies in $\underline{M}^{\Box|G/H|}(H/K)$ and is universally determined by G. Thus, $\phi_K^{\Box|G/H|}(F(x)) = F(\phi_K(x))$ and Equation 2.2.2 holds.

Lastly we will show that Φ_G commutes with the restriction map res_H^G by showing that

$$res_H^G \Phi_G(tr_H^G(x)) \quad = \quad \Phi_H res_H^G(tr_H^G(x)) \tag{2.2.3}$$

for all $tr_H^G(x)$ in $Im(tr_H^G)$, and

$$res_H^G \Phi_G(N(a)) \quad = \quad \Phi_H res_H^G(N(a)) \tag{2.2.4}$$

for all generators $N(a)$ in the $\mathbb{Z}\{\underline{M}(H/H)\}$-summand of $(N_H^G \underline{M})(G/G)$.

Equation 2.2.3 holds because

$$\begin{aligned} res_H^G \Phi_G(tr_H^G(x)) &= res_H^G tr_H^G(\phi_H^{\Box|G/H|}(x)) \\ &= \sum_{\gamma^t \in W_G(H)} \gamma^t \cdot \phi_H^{\Box|G/H|}(x) \end{aligned}$$

and

$$\begin{aligned} \Phi_H res_H^G(tr_H^G(x)) &= \phi_H^{\Box|G/H|} \left(\sum_{\gamma^t \in W_G(H)} \gamma^t \cdot x \right) \\ &= \sum_{\gamma^t \in W_G(H)} \gamma^t \cdot \phi_H^{\Box|G/H|}(x). \end{aligned}$$

Verifying Equation 2.2.4 is also straightforward:

$$\begin{aligned} res^G_H \Phi_G(N(a)) &= res^G_H(N(\phi_H(a))) \\ &= \phi_H(a)^{\otimes|G/H|} \\ &= \phi^{\Box|G/H|}(a^{\otimes|G/H|}) \\ &= \Phi_H(a^{\otimes|G/H|}) \\ &= \Phi_H res^G_H(N(a)) \end{aligned}$$

Therefore, the map $N^G_H : \mathcal{M}ack_H \to \mathcal{M}ack_G$ is a functor. $\square$

Next we prove that the norm functor N^G_H is strong symmetric monoidal.

Theorem 2.2.4. *The functor N^G_H is strong symmetric monoidal.*

Proof. Given H-Mackey functors $\underline{M}$ and $\underline{L}$ we will define an isomorphism

$$\Psi : N^G_H\underline{M} \Box N^G_H\underline{L} \to N^G_H(\underline{M} \Box \underline{L}).$$

Hence, we will need to define isomorphisms

$$\Psi_K : (N^G_H\underline{M} \Box N^G_H\underline{L})(G/K) \to N^G_H(\underline{M} \Box \underline{L})(G/K)$$

for all subgroups K of G, and these isomorphisms must commute with the appropriate restriction and transfer maps.

To define Ψ_K for K a subgroup of H we must first unpack $(N^G_H\underline{M} \Box N^G_H\underline{L})(G/K)$ and $(N^G_H(\underline{M} \Box \underline{L}))(G/K)$. Since

$$(N^G_H\underline{M})(G/K) = \underline{M}^{\Box|G/H|}(H/K) \text{ and } (N^G_H\underline{L})(G/K) = \underline{L}^{\Box|G/H|}(H/K)$$

it follows that

$$(N_H^G\underline{M} \square N_H^G\underline{L})(G/K) = (\underline{M}^{\square|G/H|} \square \underline{L}^{\square|G/H|})(H/K).$$

If $m_1 \otimes \cdots \otimes m_{|G/H|} \otimes l_1 \otimes \cdots \otimes l_{|G/H|}$ is a simple tensor in $(\underline{M}^{\square|G/H|} \square \underline{L}^{\square|G/H|})(H/K)$ and γ is the generator of G then the Weyl group $W_G(K)$ acts on this simple tensor by

$$\gamma \cdot (m_1 \otimes \cdots \otimes m_p \otimes l_1 \otimes \cdots \otimes l_p) =$$
$$(\gamma^p \cdot m_p) \otimes m_1 \otimes \cdots \otimes m_{p-1} \otimes (\gamma^p \cdot l_p) \otimes l_1 \otimes \cdots \otimes l_{p-1}$$

where γ^p is the generator $W_H(K)$. Further, $N_H^G(\underline{M}\square\underline{L})(G/K) = (\underline{M}\square\underline{L})^{\square|G/H|}(H/K)$. A simple tensor of this group is

$$m_1 \otimes l_1 \otimes m_2 \otimes l_2 \otimes \cdots \otimes m_p \otimes l_p$$

and has $W_G(K)$-action

$$\gamma \cdot (m_1 \otimes l_1 \otimes m_2 \otimes l_2 \otimes \cdots \otimes m_p \otimes l_p) =$$
$$(\gamma^p \cdot (m_p \otimes l_p)) \otimes m_1 \otimes l_1 \otimes m_2 \otimes l_2 \otimes \cdots \otimes m_{p-1} \otimes l_{p-1}.$$

Because the box product is the symmetric monoidal product in $\mathcal{M}ack_G$, we can let

$$\Psi_K : (\underline{M}^{\square|G/H|} \square \underline{L}^{\square|G/H|})(H/K) \to (\underline{M} \square \underline{L})^{\square|G/H|}(H/K)$$

be the natural isomorphism

$$m_1 \otimes \cdots \otimes m_{|G/H|} \otimes l_1 \otimes \cdots \otimes l_{|G/H|} \mapsto m_1 \otimes l_1 \otimes \cdots \otimes m_{|G/H|} \otimes l_{|G/H|}$$

that rearranges the box product factors.

It remains to define an isomorphism

$$\Psi_G : (N_H^G \underline{M} \,\square\, N_H^G \underline{L})(G/G) \to N_H^G(\underline{M} \,\square\, \underline{L})(G/G)$$

such that $res_H^G \Psi_G = \Psi_H res_H^G$ and $tr_H^G \Psi_H = \Psi_G tr_H^G$. Before we build Ψ_G we must reexamine $(N_H^G \underline{M} \,\square\, N_H^G \underline{L})(G/G)$ and $N_H^G(\underline{M} \,\square\, \underline{L})(G/G)$. First,

$$\begin{aligned}&(N_H^G \underline{M} \,\square\, N_H^G \underline{L})(G/G) = \\ &\qquad \Big((N_H^G \underline{M})(G/G) \otimes (N_H^G \underline{L})(G/G) \oplus (\underline{M}^{\square |G/H|} \,\square\, \underline{L}^{\square |G/H|})(H/H)/_{W_G(H)}\Big)\,/_{FR}.\end{aligned}$$

Since $(\underline{M}^{\square |G/H|} \,\square\, \underline{L}^{\square |G/H|})(H/H)/_{W_G(H)}$ is the image of the transfer map $tr_H^G : (N_H^G \underline{M} \,\square\, N_H^G \underline{L})(G/H) \to (N_H^G \underline{M} \,\square\, N_H^G \underline{L})(G/G)$, we will denote this summand by $Im(tr_H^G)$. To prevent notational confusion we will let T_M denote the image of tr_H^G in $(N_H^G \underline{M})(G/G)$ and T_L denote the image of tr_H^G in $(N_H^G \underline{L})(G/G)$. Hence, $(N_H^G \underline{M} \,\square\, N_H^G \underline{L})(G/G)$ becomes

$$\big(\left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus T_M\right)/_{TR_M} \otimes \left(\mathbb{Z}\{\underline{L}(H/H)\} \oplus T_L\right)/_{TR_L} \oplus Im(tr_H^G)\big)/_{FR},$$

but we can simplify this further. Frobenius reciprocity identifies all elements of T_M and T_L with elements in $Im(tr_H^G)$. Then consider an element of the form $(N(a+b) - N(a) - N(b)) \otimes \vec{y}$ in

$$\left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus T_M\right)/_{TR_M} \otimes \left(\mathbb{Z}\{\underline{L}(H/H)\} \oplus T_L\right)/_{TR_L}.$$

We can assume that $\vec{y}$ is a generator $N(y)$ in $\mathbb{Z}\{\underline{L}(H/H)\}$ because all elements of T_M and T_L are identified with elements in $Im(tr_H^G)$. If we combine the relations induced

by the submodule TR_M with Frobenius reciprocity then for all a and b in $\underline{M}(H/H)$ and y in $\underline{L}(H/H)$

$$\begin{aligned}[N(a+b)-N(a)-N(b)]\otimes N(y) &= tr_H^G(g(a,b))\otimes N(y)\\ &= tr_H^G[g(a,b)\otimes res_H^G(N(y))]\\ &= tr_H^G[g(a,b)\otimes y^{\otimes|G/H|}].\end{aligned}$$

Similarly, given a generator $N(z)$ in $\mathbb{Z}\{\underline{M}(H/H)\}$ and $N(a+b)-N(a)-N(b)$ in $\mathbb{Z}\{\underline{L}(H/H)\}$ for any a and b in $\underline{L}(H/H)$ we have

$$N(z)\otimes[N(a+b)-N(a)-N(b)] = tr_H^G[z^{\otimes|G/H|}\otimes g(a,b)].$$

Moreover, we can identify all generators of the form $N(tr_K^H(x))$ for any subgroup K of H in both $\mathbb{Z}\{\underline{M}(H/H)\}$ and $\mathbb{Z}\{\underline{L}(H/H)\}$ with elements in $Im(tr_H^G)$. Therefore,

$$(N_H^G\underline{M}\,\square\, N_H^G\underline{L})(G/G)\cong$$
$$\big(\mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)}\}\otimes\mathbb{Z}\{\underline{L}(H/H)/_{Im(tr)}\}\oplus Im(tr_H^G)\big)/_{ftr}$$

where ftr is the submodule generated by

$$[N(a+b)-N(a)-N(b)]\otimes N(y) - tr_H^G\left(g(a,b)\otimes y^{\otimes|G/H|}\right)$$

and

$$N(z)\otimes[N(c+d)-N(c)-N(d)] - tr_H^G\left(z^{\otimes|G/H|}\otimes g(c,d)\right)$$

for all a, b, and z in $\underline{M}(H/H)$ and c, d, and y in $\underline{L}(H/H)$.

Finally, the tensor product $\mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)}\}\otimes\mathbb{Z}\{\underline{L}(H/H)/_{Im(tr)}\}$ is isomorphic to $\mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)} \times \underline{L}(H/H)/_{Im(tr)}\}$. Hence,

$$(N_H^G\underline{M} \,\Box\, N_H^G\underline{L})(G/G) \cong$$

$$\big(Z\{\underline{M}(H/H)/_{Im(tr)} \times \underline{L}(H/H)/_{Im(tr)}\} \oplus Im(tr_H^G)\big)/_{ftr}$$

and the submodule ftr is generated by

$$N((a+b)\times y) - N(a\times y) - N(b\times y) - tr_H^G\left(g(a,b)\otimes y^{\otimes|G/H|}\right)$$

and

$$N(z\times(c+d)) - N(z\times c) - N(z\times d) - tr_H^G\left(z^{\otimes|G/H|}\otimes g(c,d)\right)$$

for all a, b, and z in $\underline{M}(H/H)$ and c, d, and y in $\underline{L}(H/H)$.

Now consider $N_H^G(\underline{M} \,\Box\, \underline{L})(G/G)$. We have define this group by

$$N_H^G(\underline{M} \,\Box\, \underline{L})(G/G) =$$

$$\big(\mathbb{Z}\{(\underline{M} \,\Box\, \underline{L})(H/H)\} \oplus (\underline{M} \,\Box\, \underline{L})^{\Box|G/H|}(H/H)/_{W_G(H)}\big)/_{TR}.$$

The summand $(\underline{M} \,\Box\, \underline{L})^{\Box|G/H|}(H/H)/_{W_G(H)}$ is the image of the transfer map tr_H^G, and so we denote it by $T_{M\Box L}$. Then since

$$(\underline{M} \,\Box\, \underline{L})(H/H) = (\underline{M}(H/H)\otimes\underline{L}(H/H)\oplus Im(tr))/_{FR}$$

it follows that

$$N_H^G(\underline{M} \,\Box\, \underline{L})(G/G) =$$

$$\big(\mathbb{Z}\{(\underline{M}(H/H)\otimes\underline{L}(H/H)\oplus Im(tr))/_{FR}\} \oplus T_{M\Box L}\big)/_{TR}.$$

But, by Frobenius reciprocity all elements of the form $a \otimes tr(b)$ and $tr(a) \otimes b$ in $\underline{M}(H/H) \otimes \underline{L}(H/H)$ are identified with elements in $Im(tr)$, and by Tambara reciprocity all elements of $Im(tr)$ are identified with elements of $T_{M\Box L}$. Therefore,

$$N_H^G(\underline{M} \Box \underline{L})(G/G) \cong$$

$$\big(\mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)} \otimes \underline{L}(H/H)/_{Im(tr)}\} \oplus T_{M\Box L}\big)/_{ftr'}$$

where ftr' is the submodule generated by the elements

$$N(a \otimes b + y \otimes z) - N(a \otimes b) - N(y \otimes z) - tr_H^G(g(a \otimes b, y \otimes z))$$

for all $a \otimes b$ and $y \otimes z$ in $\underline{M}(H/H)/_{Im(tr)} \otimes \underline{L}(H/H)/_{Im(tr)}$.

Hence, we will define a map $\Psi_G : (N_H^G\underline{M} \Box N_H^G\underline{L})(G/G) \to N_H^G(\underline{M} \Box \underline{L})(G/G)$ by defining two maps

$$\psi : Im(tr_H^G) \to T_{M\Box L}$$

$$\psi' : \mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)} \times \underline{L}(H/H)/_{Im(tr)}\} \to \mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)} \otimes \underline{L}(H/H)/_{Im(tr)}\}$$

such that the following relations hold.

$$\begin{aligned} \psi'(N((a+b) \times y) - N(a \times y) - N(b \times y)) &= \psi\left(tr_H^G(g(a,b) \otimes y^{\otimes|G/H|})\right) \\ \psi'(N(z \times (c+d)) - N(z \times c) - N(z \times d)) &= \psi\left(tr_H^G(z^{\otimes|G/H|} \otimes g(c,d))\right) \end{aligned}$$

Since the submodule $Im_H^G(tr) = (\underline{M}^{\Box|G/H|} \Box \underline{L}^{\Box|G/H|})(H/H)/_{W_G(H)}$ and $T_{M\Box L} = (\underline{M} \Box \underline{L})^{\Box|G/H|}(H/H)/_{W_G(H)}$, we can define the map ψ to be the isomorphism induced from the isomorphism

$$\Psi_H : (\underline{M}^{\Box|G/H|} \Box \underline{L}^{\Box|G/H|})(H/H) \xrightarrow{\cong} (\underline{M} \Box \underline{L})^{\Box|G/H|}(H/H).$$

Further, define ψ' by $\psi'(N(m \times l)) = N(m \otimes l)$. Then given a and b in $\underline{M}(H/H)$ and y in $\underline{L}(H/H)$

$$\begin{aligned} &\psi'(N((a+b)\times y) - N(a \times y) - N(b \times y)) \\ &= N((a+b)\otimes y) - N(a\otimes y) - N(b \otimes y) \\ &= N(a\otimes y + b\otimes y) - N(a\otimes y) - N(b\otimes y) \\ &= tr_H^G\left(g(a\otimes y, b\otimes y)\right), \end{aligned}$$

but $\psi(tr_H^G(g(a,b)\otimes y^{\otimes|G/H|})) = tr_H^G(g(a\otimes y, b\otimes y))$. Similarly,

$$\psi'(N(z\times(c+d)) - N(z\times c) - N(z\times d)) = \psi(tr_H^G(z^{\otimes|G/H|}\otimes g(c,d)))$$

for all z in $\underline{M}(H/H)$ and c and d in $\underline{L}(H/H)$. Moreover, the construction of Ψ_G forces $res_H^G\Psi_G$ to equal $\Psi_H res_H^G$ and $tr_H^G\Psi_H$ to equal $\Psi_G tr_H^G$.

Finally, to show that this well-define map is an isomorphism we will create the following diagram of short exact sequences such that the maps α and β are isomorphisms. Then, by the Five Lemma, Ψ_G will an isomorphism as well.

$$\begin{tikzcd} 0 \arrow[r] & Im(tr_H^G) \arrow[r] \arrow[d, "\alpha"] & (N_H^G\underline{M}\,\Box\, N_H^G\underline{L})(G/G) \arrow[r] \arrow[d, "\Psi_G"] & (N_H^G\underline{M}\,\Box\, N_H^G\underline{L})(G/G)/_{Im(tr_H^G)} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & T_{M\Box L} \arrow[r] & N_H^G(\underline{M}\,\Box\,\underline{L})(G/G) \arrow[r] & N_H^G(\underline{M}\,\Box\,\underline{L})(G/G)/_{T_{M\Box L}} \arrow[r] & 0 \end{tikzcd}$$

We will let the map α be the isomorphism ψ.

Before defining the map β we will simplify $(N_H^G\underline{M}\,\Box\, N_H^G\underline{L})(G/G)/_{Im(tr_H^G)}$ and $N_H^G(\underline{M}\,\Box\,\underline{L})(G/G)/_{T_{M\Box L}}$. To begin, the group

$$(N_H^G\underline{M}\,\Box\, N_H^G\underline{L})(G/G)/_{Im(tr_H^G)} = \mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)} \times \underline{L}(H/H)/_{Im(tr)}\}/_Q$$

where Q is the submodule generated by the elements $N((a+b)\times y)-N(a\times y)-N(b\times y)$ and $N(z\times(c+d))-N(z\times c)-N(z\times d)$ for all a, b, and z in $\underline{M}(H/H)/_{Im(tr_H^G)}$ and c, d, and y in $\underline{L}(H/H)/_{Im(tr_H^G)}$. The group

$$N_H^G(\underline{M}\,\Box\,\underline{L})(G/G)/_{T_{M\Box L}}=\mathbb{Z}\{\underline{M}(H/H)/_{Im(tr)}\otimes\underline{L}(H/H)_{/Im(tr)}\}/_{Q'}$$

where Q' is the submodule generated by $N(a\otimes b+x\otimes y)-N(a\otimes b)-N(x\otimes y)$ for all $a\otimes b$ and $x\otimes y$ in $\underline{M}(H/H)/_{Im(tr)}\otimes\underline{L}(H/H)_{/Im(tr)}$. Hence, $N_H^G(\underline{M}\,\Box\,\underline{L})(G/G)/_{T_{M\Box L}}$ is isomorphic to $\underline{M}(H/H)/_{Im(tr)}\otimes\underline{L}(H/H)/_{Im(tr)}$. We define

$$\beta:(N_H^G\underline{M}\,\Box\,N_H^G\underline{L})(G/G)/_{Im(tr_H^G)}\to N_H^G(\underline{M}\,\Box\,\underline{L})(G/G)/_{T_{M\Box L}}$$

by

$$\beta(N(m\times l))=\psi'(N(m\times l)).$$

But, $\psi'(N(m\times l))=N(m\otimes l)\cong m\otimes l$, and so the map β is an isomorphism by definition of the tensor product. Further, it is clear by construction of Ψ_G that the diagram of short exact sequences commutes.. Therefore, by the five lemma, the map Ψ_G is an isomorphism. $\Box$

2.2.2 Part 2: H is Any Subgroup of G

We will now extend the definition of the norm functor $N_H^G:\mathcal{M}ack_H\to\mathcal{M}ack_G$ to any subgroup H of G. Throughout this section let γ be the generator of G, H be C_{p^k} for some $k<n-1$, and $\underline{M}$ be an H-Mackey functor. The restriction $i_H^*N_H^G\underline{M}$ must still equal $\underline{M}^{\Box|G/H|}$, and hence, if H' is a subgroup of H then $(N_H^G\underline{M})(G/H')=$

$\underline{M}^{\square|G/H|}(H/H')$. The generator γ of $W_G(H')$ acts on simple tensors of $\underline{M}^{\square|G/H|}(H/H')$ by

$$\gamma \cdot (m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p^{n-k}-1}}) = (\gamma^{p^{n-k}} \cdot m_{\gamma^{p^{n-k}-1}}) \otimes m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p^{n-k}-2}}$$

where $\gamma^{p^{n-k}}$ is the generator of $W_H(H')$.

Remark 2.2.6. As we did in Remark 2.2.2 for any subgroup H' of H we can write a simple tensor

$$m_e \otimes m_\gamma \otimes \cdots \otimes m_{p^{n-k}-1}$$

in $\underline{M}^{\square|G/H|}(H/H')$ as a product over the $W_G(H')$-action:

$$(m_e \otimes 1^{\otimes p^{n-k}-1})\gamma(m_\gamma \otimes 1^{\otimes p^{n-k}-1}) \cdots \gamma^{p^{n-k}-1}(m_{\gamma^{p^{n-k}-1}} \otimes 1^{\otimes p^{n-k}-1}).$$

This concept remains valid even when considering the Cartesian product $\underline{M}(H/H)^{\times|G/K|}$ for a subgroup K of G such that $K = C_{p^i}$ and $H < K < G$. Letting $\eta = p^{n-i}$, this Cartesian product supports a $W_G(K)$-action given by

$$\gamma(m_e \times m_\gamma \times \cdots \times m_{\gamma^{\eta-1}}) = m_{\gamma^{\eta-1}} \times m_e \times m_\gamma \times \cdots \times m_{\gamma^{\eta-2}}.$$

Thus, we can write an element

$$m_e \times m_\gamma \times \cdots \times m_{\gamma^{\eta-1}}$$

in $\underline{M}(H/H)^{\times|G/K|}$ as the following product over the $W_G(K)$-action.

$$(m_e \times 1^{\times(\eta-1)})\gamma(m_\gamma \times 1^{\times(\eta-1)}) \cdots \gamma^{\eta-1}(m_{\gamma^{(\eta-1)}} \times 1^{\times(\eta-1)})$$

If K' is a subgroup C_{p^r} such that $H < K' < K$ then we can embed the element $m_e \times \cdots \times m_{\gamma^{\eta-1}}$ in $\underline{M}(H/H)^{\times|G/K'|}$ by

$$m_e \times \cdots \times m_{\gamma^{\eta-1}} \mapsto m_e \times \cdots \times m_{\gamma^{\eta-1}} \times 1^{\times|G/K'|-\eta}.$$

Moreover, we can consider an element

$$m_e \times m_\gamma \times \cdots \times m_{\gamma^{|G/K'|-1}}$$

in $\underline{M}(H/H)^{\times|G/K'|}$ as a product over the $W_K(K')$-action since we can write

$$\begin{aligned}
&m_e \times m_\gamma \times \cdots \times m_{\gamma^{|G/K'|-1}} \\
&\quad = (m_e \times \cdots \times m_{\gamma^{\eta-1}}) \times (m_{\gamma^\eta} \times \cdots \times m_{\gamma^{2\eta-1}}) \times \cdots \times (m_{\gamma^{(p^{i-r}-1)\eta}} \times \cdots \times m_{\gamma^{|G/K'|-1}}) \\
&\quad = (m_e \times \cdots \times m_{\gamma^{\eta-1}} \times 1^{\times|G/K'|-\eta})\gamma^\eta(m_{\gamma^\eta} \times \cdots \times m_{\gamma^{2\eta-1}} \times 1^{\times|G/K'|-\eta}) \cdots \\
&\qquad \cdots \gamma^{(p^{i-r}-1)\eta}(m_{\gamma^{(p^{i-r}-1)\eta}} \times \cdots \times m_{\gamma^{|G/K'|-1}} \times 1^{\times|G/K'|-\eta}).
\end{aligned}$$

Thus, given elements

$$A = m_e \times \cdots \times a_{\gamma^j} \times \cdots \times m_{\gamma^{\eta-1}} \times 1^{\times|G/K'|-\eta}$$

and

$$B = m_e \times \cdots \times b_{\gamma^j} \times \cdots \times m_{\gamma^{\eta-1}} \times 1^{\times|G/K'|-\eta},$$

in $\underline{M}(H/H)^{\times|G/K'|}$, we can make sense of evaluating the monomials $(\vec{ab})_i^{K'}$ of Equation 1.4.2 at A, B, and their $W_K(K')$-conjugates. Every monomial $(\vec{ab})_i^{K'}$ is a product of the elements a, b, and their $W_K(K')$-conjugates and consists only of elements $\gamma^s a$ and $\gamma^s b$. They do not contain elements of the form $\gamma^s a^m$ and $\gamma^s b^m$ for $m > 1$. Therefore,

when we evaluate $(\vec{ab})_i^{K'}$ at A, B, and their $W_K(K')$-conjugates we obtain a single product in $\underline{M}(H/H)^{\times|G/K'|}$.

For example, in a C_{16}-Tambara functor, Equation 1.4.2 for $N_{C_2}^{C_8}(a+b)$ is

$$N_{C_2}^{C_8}(a+b) = \tag{2.2.5}$$
$$N_{C_2}^{C_8}(a) + N_{C_2}^{C_8}(b) + tr_{C_2}^{C_8}(g_{C_2}(a,b)) + tr_{C_4}^{C_8}\left(N_{C_2}^{C_4}((\vec{ab})^{C_4})\right)$$

where $(\vec{ab})^{C_4} = a\gamma^2 b$. Given a C_2-Mackey functor $\underline{M}$, consider the elements $a_e \times m_\gamma$ and $b_e \times m_\gamma$ in $\underline{M}(C_2/C_2)^{\times|C_{16}/C_8|}$. We can embed these elements in $\underline{M}(C_2/C_2)^{\times|C_{16}/C_4|}$ by writing them as $a_e \times m_\gamma \times 1 \times 1$ and $b_e \times m_\gamma \times 1 \times 1$. Evaluating $a\gamma^2 b$ at these elements gives

$$\begin{aligned}(a_e \times m_\gamma \times 1 \times 1)\gamma^2(b_e \times m_\gamma \times 1 \times 1) &= (a_e \times m_\gamma \times 1 \times 1)(1 \times 1 \times b_e \times m_\gamma) \\ &= a_e \times m_\gamma \times b_e \times m_\gamma.\end{aligned}$$

Similarly, we can embed an element $m_e \times \cdots \times m_{\gamma^{\eta-1}}$ of $\underline{M}(H/H)^{\times|G/K|}$ in $\underline{M}(H/H)^{\otimes|G/H|}$ by

$$m_e \times \cdots \times m_{\gamma^{\eta-1}} \mapsto m_e \otimes \cdots \otimes m_{\gamma^{\eta-1}} \otimes 1^{|G/H|-\eta},$$

and we can write any simple tensor

$$m_e \otimes \cdots \otimes m_{\gamma^{\eta-1}} \otimes m_{\gamma^\eta} \otimes \cdots \otimes m_{\gamma^{|G/H|-1}}$$

in $\underline{M}(H/H)^{\otimes|G/H|}$ as a product over the $W_K(H)$-action. Hence, it makes sense to evaluate the polynomial $g_H(a,b)$ of Equation 1.4.2 at

$$m_e \otimes \cdots \otimes m_{\gamma^{j-1}} \otimes a_{\gamma^j} \otimes m_{\gamma^{j+1}} \otimes \cdots \otimes m_{\gamma^{\eta-1}} \otimes 1^{\otimes|G/H|-\eta},$$

$$m_e \otimes \cdots \otimes m_{\gamma^{j-1}} \otimes b_{\gamma^j} \otimes m_{\gamma^{j+1}} \otimes \cdots \otimes m_{\gamma^{\eta-1}} \otimes 1^{\otimes|G/H|-\eta},$$

and their $W_K(H)$-conjugates.

For example, a monomial in $g_{C_2}(a,b)$ of Equation 2.2.5 is $a\gamma^2 a\gamma^4 a\gamma^6 b$. The elements $a_e \times m_\gamma$ and $b_e \times m_\gamma$ of $\underline{M}(C_2/C_2)^{\times|C_{16}/C_8|}$ embed into $\underline{M}(C_2/C_2)^{\otimes|C_{16}/C_2|}$ as $a_e \otimes m_\gamma \otimes 1^{\otimes 6}$ and $b_e \otimes m_\gamma \otimes 1^{\otimes 6}$, respectively. Thus, we can evaluate $a\gamma^2 a\gamma^4 a\gamma^6 b$ at these elements and their $W_{C_8}(C_4)$-conjugates:

$$\begin{aligned}
&(a_e \otimes m_\gamma \otimes 1^{\otimes 6})\gamma^2(a_e \otimes m_\gamma \otimes 1^{\otimes 6})\gamma^4(a_e \otimes m_\gamma \otimes 1^{\otimes 6})\gamma^6(b_e \otimes m_\gamma \otimes 1^{\otimes 6}) \\
&= (a_e \otimes m_\gamma \otimes 1^{\otimes 6})(1^{\otimes 2} \otimes a_e \otimes m_\gamma \otimes 1^{\otimes 4})(1^{\otimes 4} \otimes a_e \otimes m_\gamma \otimes 1^{\otimes 2})(1^{\otimes 6} \otimes b_e \otimes m_\gamma) \\
&= a_e \otimes m_\gamma \otimes a_e \otimes m_\gamma \otimes a_e \otimes m_\gamma \otimes b_e \otimes m_\gamma.
\end{aligned}$$

Furthermore, given an element

$$m_e \times \cdots \times m_{\gamma^{j-1}} \times tr_{H'}^{H}(x)_{\gamma^j} \times m_{\gamma^{j+1}} \times \cdots \times m_{\gamma^{\eta-1}}$$

in $\underline{M}(H/H)^{\times|G/K|}$ we can consider the corresponding element

$$res_{H'}^{H}(m_e) \otimes \cdots \otimes res_{H'}^{H}(m_{\gamma^{j-1}}) \otimes x_{\gamma^j} \otimes res_{H'}^{H}(m_{\gamma^{j+1}}) \otimes \cdots \otimes res_{H'}^{H}(m_{\gamma^{\eta-1}}) \otimes 1^{\otimes|G/H|-\eta}$$

in $\underline{M}(H/H')^{\otimes|G/H|}$. Since the polynomial $f(x)$ of Equation 1.4.6 consists only of monomials containing single powered elements it makes sense to evaluate $f(x)$ at

$$res_{H'}^{H}(m_e) \otimes \cdots \otimes x_{\gamma^j} \otimes \cdots \otimes res_{H'}^{H}(m_{\gamma^{\eta-1}}) \otimes 1^{\otimes|G/H|-\eta}$$

and its $W_K(H')$-conjugates to obtain a simple tensor in $\underline{M}^{\otimes|G/H|}(H/H)$.

For example, in a C_{16}-Tambara functor, Equation 1.4.6 for $N_{C_2}^{C_8}(tr_e^{C_2}(x))$ is

$$N_{C_2}^{C_8}(tr_e^{C_2}(x)) = tr_e^{C_8}(f(x)),$$

and one of the monomials in $f(x)$ is $x\gamma^2 x\gamma^{12} x\gamma^6 x$. Consider the element $tr_e^{C_2}(x) \times m_\gamma$ in $\underline{M}(C_2/C_2)^{\times|C_{16}/C_8|}$. We can evaluate the above monomial at $x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 6}$ and its $W_{C_8}(e)$-conjugates.

$$\begin{aligned}
& \left(x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 6}\right) \gamma^2 \left(x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 6}\right) \\
& \quad \gamma^{12} \left(x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 6}\right) \gamma^6 \left(x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 6}\right) \\
= \; & \left(x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 6}\right) \left(1^{\otimes 2} \otimes x \otimes res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 4}\right) \\
& \left(1^{\otimes 4} \otimes \gamma^8 x \otimes \gamma^8 res_e^{C_2}(m_\gamma) \otimes 1^{\otimes 2}\right) \left(1^{\otimes 6} \otimes x \otimes res_e^{C_2}(m_\gamma)\right) \\
= \; & x \otimes res_e^{C_2}(m_\gamma) \otimes x \otimes res_e^{C_2}(m_\gamma) \otimes \gamma^8 x \otimes \gamma^8 res_e^{C_2}(m_\gamma) \otimes x \otimes res_e^{C_2}(m_\gamma)
\end{aligned}$$

We now provide a complete definition of the norm functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ for a nonmaximal subgroup H of G. We will then prove that for all subgroups H and K of G such that $H < K < G$ the map N_H^G is isomorphic to the composition $N_K^G N_H^K$. The proof that N_H^G is, in fact, a symmetric monoidal functor for all H is an easy corollary of this result.

Definition 2.2.5. Given an H-Mackey functor $\underline{M}$ define the norm functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ as follows. If H' is a subgroup of H then define

$$(N_H^G \underline{M})(G/H') := \underline{M}^{\square|G/H|}(H/H').$$

The $W_G(H')$-action on a simple tensor is given by

$$\gamma \cdot (m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p^{n-k}-1}}) = (\gamma^{p^{n-k}} \cdot m_{\gamma^{p^{n-k}-1}}) \otimes m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p^{n-k}-2}}$$

where $\gamma^{p^{n-k}}$ as the generator of $W_H(H')$. If $H' < H'' < H$ then the restriction map $res_{H'}^{H''}$ and the transfer map $tr_{H'}^{H''}$ are the restriction and transfer maps given in

Definition 1.2.1. If K is any subgroup C_{p^i} of G such that $k < i \leq n$ then define

$$(N_H^G \underline{M})(G/K) := \left(\mathbb{Z}\{\underline{M}(H/H)^{\times |G/K|}\} \oplus \left((N_H^G \underline{M})(G/C_{p^{i-1}}) \right) /_{W_K(C_{p^{i-1}})} \right) /_{FTR}.$$

1. For $m_e \times m_\gamma \times \cdots \times m_{\gamma^{p^{n-i}-1}}$ in $\underline{M}(H/H)^{\times |G/H|}$ let

$$N(m_e \times m_\gamma \times \cdots \times m_{\gamma^{p^{n-i}-1}})$$

denote the corresponding generator of $\mathbb{Z}\{\underline{M}(H/H)^{\times |G/K|}\}$. The generator γ of $W_G(K)$ acts on this generator by

$$\gamma \cdot N(m_e \times m_\gamma \times \cdots \times m_{\gamma^{p^{n-i}-1}}) = N(m_{\gamma^{p^{n-i}-1}} \times m_e \times m_\gamma \times \cdots \times m_{\gamma^{p^{n-i}-2}}).$$

2. The transfer map $tr^K_{C_{p^{i-1}}}$ is the canonical quotient map onto $\left((N_H^G \underline{M})(G/C_{p^{i-1}}) \right) /_{W_K(C_{p^{i-1}})}$. We will refer to $\left((N_H^G \underline{M})(G/C_{p^{i-1}}) \right) /_{W_K(C_{p^{i-1}})}$ as $Im(tr^K_{C_{p^{i-1}}})$ and an element in this summand as $tr^K_{C_{p^j}}(\vec{x})$ for some $\vec{x}$ in $(N_H^G \underline{M})(G/C_{p^{i-1}})$.

3. The restriction map $res^K_{C_{p^{i-1}}}$ is given by

$$res^K_{C_{p^{i-1}}}(tr^K_{C_{p^j}}(\vec{x})) = \sum_{\gamma^s \in W_K(C_{p^{i-1}})} \gamma^s \cdot \vec{x}$$

and

$$\begin{aligned} & res^K_{C_{p^{i-1}}}(N(m_e \times \cdots \times m_{\gamma^{p^{n-i}-1}})) \\ = & \begin{cases} N\left((m_e \times \cdots \times m_{\gamma^{p^{n-i}-1}})^{\times |K/C_{p^{i-1}}|} \right) & \text{if } C_{p^{i-1}} > H \\ (m_e \otimes \cdots \otimes m_{\gamma^{p^{n-i}-1}})^{\otimes |K/C_{p^{i-1}}|} & \text{if } C_{p^{i-1}} = H \end{cases} \end{aligned}$$

4. The submodule FTR is generated by elements of the form

$$\begin{aligned} & N(m_e \times m_\gamma \times \cdots \times (a_{\gamma^j} + b_{\gamma^j}) \times \cdots \times m_{\gamma^{p^{n-i}-1}}) \\ - \quad & N(m_e \times m_\gamma \times \cdots \times a_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}}) \\ - \quad & N(m_e \times m_\gamma \times \cdots \times b_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}}) \\ - \quad & \sum_{H<K'<K} tr_{K'}^{K} \left(\sum_i N\left((\overrightarrow{a_{\gamma^j} b_{\gamma^j}})_i^{K'} \right) \right) - tr_H^K \left(g(a_{\gamma^j}, b_{\gamma^j}) \right) \end{aligned} \tag{2.2.6}$$

and

$$N(m_e \times m_\gamma \times \cdots \times tr_{H'}^{H}(x)_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}}) - tr_{H'}^{K} \left(f(x_{\gamma^j}) \right) \tag{2.2.7}$$

for all a and b in $\underline{M}(H/H)$, x in $\underline{M}(H/H')$, γ^j such that $e \leq \gamma^j \leq \gamma^{p^{n-i}-1}$ and subgroups H' of H. The term $(\overrightarrow{a_{\gamma^j} b_{\gamma^j}})_i^{K'}$ is the monomial $(\vec{ab})_i^{K'}$ of Equation 1.4.2 evaluated at

$$m_e \times m_\gamma \times \cdots \times a_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}} \times 1^{\times |G/K'|-|G/K|},$$

$$m_e \times m_\gamma \times \cdots \times b_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}} \times 1^{\times |G/K'|-|G/K|},$$

and their $W_K(K')$-conjugates. The polynomial $g(a_{\gamma^j}, b_{\gamma^j})$ is the polynomial $g_H(a,b)$ of Equation 1.4.2 evaluated at

$$m_e \otimes \cdots \otimes a_{\gamma^j} \otimes \cdots \otimes m_{\gamma^{\eta-1}} \otimes 1^{\otimes |G/H|-|G/K|},$$

$$m_e \otimes \cdots \otimes b_{\gamma^j} \otimes \cdots \otimes m_{\gamma^{\eta-1}} \otimes 1^{\otimes |G/H|-|G/K|},$$

and their $W_K(H)$-conjugates as discussed in Remark 2.2.6. The polynomial $f(x_{\gamma^j})$ is the polynomial $f(x)$ of Equation 1.4.6 evaluated at

$$res_{H'}^{H}(m_e) \otimes res_{H'}^{H}(m_\gamma) \otimes \cdots \otimes x_{\gamma^j} \otimes \cdots \otimes res_{H'}^{H}(m_{\gamma^{p^{n-i}-1}}) \otimes 1^{\otimes |G/H|-|G/K|}$$

also discussed in Remark 2.2.6.

It is not obvious that the submodule FTR in $(N_H^G\underline{M})(G/K)$ is $W_G(K)$-equivariant. But, since γ is the generator of $W_G(K)$, in Relation 2.2.6 we have

$$\begin{aligned}\gamma \cdot \big[&N(m_e \times \cdots \times (a+b)_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}}) \\ &-N(m_e \times \cdots \times a_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}}) - N(m_e \times \cdots \times b_{\gamma^j} \times \cdots \times m_{\gamma^{p^{n-i}-1}})\big] \\ = \quad & N(m_{\gamma^{p^{n-i}-1}} \times m_e \times \cdots \times (a+b)_{\gamma^{j+1}} \times \cdots \times m_{\gamma^{p^{n-i}-2}}) \\ &-N(m_{\gamma^{p^{n-i}-1}} \times m_e \times \cdots \times a_{\gamma^{j+1}} \times \cdots \times m_{\gamma^{p^{n-i}-2}}) \\ &-N(m_{\gamma^{p^{n-i}-1}} \times m_e \times \cdots \times b_{\gamma^{j+1}} \times \cdots \times m_{\gamma^{p^{n-i}-2}}).\end{aligned}$$

Moreover, $\gamma \cdot g(a_{\gamma^j}, b_{\gamma^j})$ is the polynomial $g_H(a,b)$ of Equation 1.4.2 evaluated at

$$m_{\gamma^{p^{n-i}-1}} \otimes m_e \otimes \cdots \otimes a_{\gamma^{j+1}} \otimes \cdots \otimes m_{\gamma^{p^{n-i}-2}} \otimes 1^{\otimes |G/H|-|G/K|},$$

$$m_{\gamma^{p^{n-i}-1}} \otimes m_e \otimes \cdots \otimes b_{\gamma^{j+1}} \otimes \cdots \otimes m_{\gamma^{p^{n-i}-2}} \otimes 1^{\otimes |G/H|-|G/K|},$$

and their Weyl conjugates, and $\gamma \cdot (\overrightarrow{a_{\gamma^j} b_{\gamma^j}})_i^{K'}$ is the monomial $(\vec{ab})_i^{K'}$ of Equation 1.4.2 evaluated at

$$m_{\gamma^{p^{n-i}-1}} \times m_e \times \cdots \times a_{\gamma^{j+1}} \times \cdots \times m_{\gamma^{p^{n-i}-2}} \times 1^{\times |G/K'|-|G/K|},$$

$$m_{\gamma^{p^{n-i}-1}} \times m_e \times \cdots \times b_{\gamma^{j+1}} \times \cdots \times m_{\gamma^{p^{n-i}-2}} \times 1^{\times |G/K'|-|G/K|},$$

and their Weyl conjugates, and thus Relation 2.2.6 holds under the $W_G(K)$-action. Because $tr_{H'}^K(f(x_{\gamma^j}))$ is an element in $(N_H^G\underline{M})(G/C_{p^{i-1}})/_{W_K(C_{p^{i-1}})}$, a similar argument will show that Relation 2.2.7 is $W_G(K)$-equivariant.

Example 2.2.7. Let $\underline{M}$ be a C_2-Mackey functor. The C_8-Mackey functor $N_{C_2}^{C_8}\underline{M}$ is shown in Figure 2.1. The submodule FTR_{C_4} is generated by

$$\begin{tikzcd}
\left(\mathbb{Z}\{\underline{M}(C_2/C_2)\} \oplus \overbrace{(N_H^G\underline{M})(C_8/C_4)/_{W_{C_8}(C_4)}}^{Im(tr_{C_4}^{C_8})}\right)/_{FTR_{C_8}} \arrow[d, bend right=50, "res_{C_4}^{C_8}"'] \\
\left(\mathbb{Z}\{\underline{M}(C_2/C_2) \times \underline{M}(C_2/C_2)\} \oplus \overbrace{\underline{M}^{\square 4}(C_2/C_2)/_{W_{C_4}(C_2)}}^{Im(tr_{C_2}^{C_4})}\right)/_{FTR_{C_4}} \arrow[u, bend right=50, "tr_{C_4}^{C_8}"'] \arrow[d, bend right=50, "res_{C_2}^{C_4}"'] \\
\underline{M}^{\square|C_8/C_2|}(C_2/C_2) \arrow[u, bend right=50, "tr_{C_2}^{C_4}"'] \arrow[d, bend right=50, "res_e^{C_2}"'] \\
\underline{M}(C_2/e)^{\otimes|C_8/C_2|} \arrow[u, bend right=50, "tr_e^{C_2}"']
\end{tikzcd}$$

Figure 2.1: The Mackey Functor Diagram for $N_{C_2}^{C_8}\underline{M}$

$$N((a_e + b_e) \times m_\gamma) - N(a_e + m_\gamma) - N(b_e + m_\gamma) - tr_{C_2}^{C_4}(a_e \otimes m_\gamma \otimes b_e \otimes m_\gamma),$$

$$N(m_e + (a_\gamma + b_\gamma)) - N(m_e + a_\gamma) - N(m_e + b_\gamma) - tr_{C_2}^{C_4}(m_e \otimes a_\gamma \otimes m_e \otimes b_\gamma),$$

$$N(tr_e^{C_2}(x) \otimes m_\gamma) - tr_e^{C_4}(x \otimes res_e^{C_2}(m_\gamma) \otimes x \otimes res_e^{C_2}(m_\gamma)), \text{ and}$$

$$N(m_e \otimes tr_e^{C_2}(x)) - tr_e^{C_4}(res_e^{C_2}(m_e) \otimes x \otimes res_e^{C_2}(m_e) \otimes x),$$

and FTR_{C_8} is generated by

$$
\begin{aligned}
& N(a+b) - N(a) - N(b) \\
& - \quad tr_{C_4}^{C_8}(N(a \times b)) - tr_{C_2}^{C_8}(a \otimes a \otimes a \otimes b + b \otimes b \otimes b \otimes a + a \otimes a \otimes b \otimes b), \text{ and} \\
& \quad N(tr_e^{C_2}(a)) - tr_{C_2}^{C_8}(a \otimes a \otimes a \otimes a + a \otimes \gamma^4 a \otimes a \otimes a).
\end{aligned}
$$

Further, $res_{C_4}^{C_8}(N(a)) = N(a \times a)$ and $res_{C_2}^{C_4}(N(m_e \times m_\gamma)) = m_e \otimes m_\gamma \otimes m_e \otimes m_\gamma$.

When $K = G$ Definition 2.2.5 gives

$$(N_H^G \underline{M})(G/G) = \left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus Im(tr_{C_{p^{n-1}}}^G)\right) /_{FTR}.$$

This definition is analogous to the definition of $(N_H^G \underline{M})(G/G)$ given in Definition 2.2.1, and we can still define the following map.

Definition 2.2.6. Define the map $N : \underline{M}(H/H) \to (N_H^G \underline{M})(G/G)$ by letting $N(a)$ be the corresponding generator in the free summand $\mathbb{Z}\{\underline{M}(H/H)\}$ of $(N_H^G \underline{M})(G/G)$.

To show that a G-commutative monoid has the structure of a Tambara functor we need this map $N : \underline{M}(G/H) \to (N_H^G i_H^* \underline{M})(G/G)$ to extend to the internal norm map $N_H^G : \underline{M}(G/H) \to \underline{M}(G/G)$ of $\underline{M}$. Thus, we quotient out by the FTR submodule to force the relations between the internal norm map and the transfer map described in Properties 4 and 5 of Definition 1.4.2. But, in order for a G-commutative monoid to be a Tambara functor we also need Property 2 of Definition 1.4.2 to hold. In particular, the internal norm map N_H^G must equal $N_K^G N_H^K$ whenever $H < K < G$. Therefore, we must show that the map $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ to be naturally isomorphic to the composition $N_K^G N_H^K$.

Moreover, if N_H^G is isomorphic to $N_K^G N_H^K$ for all subgroups K such that $H < K < G$ then since we have shown that N_H^G is a symmetric monoidal functor when H is maximal in G, it will follow that N_H^G is a symmetric monoidal functor for any subgroup H of G.

Theorem 2.2.7. *Suppose that $H < K < G$ and the maps $N_K^G : \mathcal{M}ack_K \to \mathcal{M}ack_G$ and $N_H^K : \mathcal{M}ack_H \to \mathcal{M}ack_K$ are as defined in Definition 2.2.5. Then the map $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ is naturally isomorphic to the composition $N_K^G N_H^K$.*

Proof. Using induction on n it will suffice to show that N_H^G is naturally isomorphic to $N_{C_{p^{n-1}}}^G N_H^{C_{p^{n-1}}}$. We will write N_m^j for $N_{C_{p^m}}^{C_{p^j}}$, and thus, given an H-Mackey functor $\underline{M}$, (since $H = C_{p^k}$,) we will construct an isomorphism $\Phi : N_{n-1}^n N_k^{n-1} \underline{M} \to N_k^n \underline{M}$ by defining isomorphisms $\Phi_K : (N_{n-1}^n N_k^{n-1} \underline{M})(G/K) \to (N_k^n \underline{M})(G/K)$ for all subgroups K of G such that each Φ_K is compatible with the appropriate restriction and transfer maps.

We will first define Φ_K for K a subgroup of H. If $K \leq H$ then

$$\begin{aligned}
(N_{n-1}^n N_k^{n-1} \underline{M})(G/K) &= (N_k^{n-1} \underline{M})^{\Box |G/C_{p^{n-1}}|}(C_{p^{n-1}}/K) \\
&= (N_k^{n-1} \underline{M}_e \Box N_k^{n-1} \underline{M}_\gamma \Box \cdots \Box N_k^{n-1} \underline{M}_{\gamma^{p-1}})(C_{p^{n-1}}/K)
\end{aligned}$$

where $N_k^{n-1} \underline{M}_{\gamma^j}$ denotes the copy of $N_k^{n-1} \underline{M}$ indexed by the element γ^j of $G/C_{p^{n-1}}$. Similarly,

$$\begin{aligned}
(N_k^{n-1} \underline{M})(C_{p^{n-1}}/K) &= \underline{M}^{\Box |C_{p^{n-1}}/H|}(H/K) \\
&= (\underline{M}_e \Box \underline{M}_{\gamma^p} \Box \cdots \Box \underline{M}_{\gamma^{p^{n-k}-p}})(H/K).
\end{aligned}$$

Hence,

$$\begin{aligned}(N^n_{n-1}N^{n-1}_k\underline{M})(G/K) &= \left(\underline{M}^{\Box|C_{p^{n-1}}/H|}\right)^{\Box|G/C_{p^{n-1}}|}(H/K)\\ &= (\underline{M}_e\Box\cdots\Box\underline{M}_{\gamma^{p^{n-k}-p}}\Box\underline{M}_\gamma\Box\cdots\Box\underline{M}_{\gamma^{p^{n-k}-p+1}}\Box\cdots\\ &\quad \cdots\Box\underline{M}_{\gamma^{p-1}}\Box\cdots\Box\underline{M}_{\gamma^{p^{n-k}-1}})(H/K),\end{aligned}$$

and we denote a simpler tensor of $(N^n_{n-1}N^{n-1}_k\underline{M})(G/K)$ by

$$\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip}}\right)\otimes\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+1}}\right)\otimes\cdots\otimes\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+p-1}}\right)$$

where $\alpha = p^{n-k-1}-1$. Then the generator γ of the Weyl group $W_G(K)$ acts on this element by

$$\begin{aligned}&\gamma\cdot\left(\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip}}\right)\otimes\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+1}}\right)\otimes\cdots\otimes\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+p-1}}\right)\right)\\ &= \left(\gamma^{p-1}\cdot\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+p-1}}\right)\otimes\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip}}\right)\otimes\cdots\otimes\left(\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+p-2}}\right).\end{aligned}$$

The element γ^{p-1} is the generator of the Weyl group $W_{C_{p^{n-1}}}(K)$, and

$$\begin{aligned}\gamma^{p-1}\cdot\bigotimes_{i=0}^{\alpha} m_{\gamma^{ip+p-1}} &= \gamma^{p-1}\cdot\left(m_{\gamma^{p-1}}\otimes\cdots\otimes m_{\gamma^{p^{n-k}-1}}\right)\\ &= (\gamma^{p^{n-k}}\cdot m_{\gamma^{p^{n-k}-1}})\otimes m_{\gamma^{p-1}}\otimes\cdots\otimes m_{\gamma^{p^{n-k}-p-1}}\end{aligned}$$

where $\gamma^{p^{n-k}}$ is the generator of $W_H(K)$.

On the other hand,

$$\begin{aligned}(N^n_k\underline{M})(G/K) &= \underline{M}^{\Box|G/H|}(H/K)\\ &= (\underline{M}_e\Box\underline{M}_\gamma\Box\cdots\Box\underline{M}_{\gamma^{p^{n-k}-2}}\Box\underline{M}_{\gamma^{p^{n-k}-1}})(H/K),\end{aligned}$$

and the $W_G(K)$-action is given by

$$\gamma \cdot (m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p^{n-k}-2}} \otimes m_{\gamma^{p^{n-k}-1}}) = \\ (\gamma^{p^{n-k}} \cdot m_{\gamma^{p^{n-k}-1}}) \otimes m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p^{n-k}-2}}.$$

Therefore, we define the map

$$\Phi_K : \left(\underline{M}^{\Box|C_{p^{n-1}}/H|}\right)^{\Box|G/C_{p^{n-1}}|}(H/K) \to \underline{M}^{\Box|G/H|}(H/K)$$

to be the isomorphism that rearranges the indices. More specifically, the map Φ_K sends

$$(\underline{M}_e \Box \cdots \Box \underline{M}_{\gamma^{p^{n-k}-p}}) \Box (\underline{M}_\gamma \Box \cdots \Box \underline{M}_{\gamma^{p^{n-k}-p+1}}) \Box \cdots \Box (\underline{M}_{\gamma^{p-1}} \Box \cdots \Box \underline{M}_{\gamma^{p^{n-k}-1}})(H/K)$$

to

$$(\underline{M}_e \Box \underline{M}_\gamma \Box \underline{M}_{\gamma^2} \Box \cdots \Box \underline{M}_{\gamma^{p^{n-k}-2}} \Box \underline{M}_{\gamma^{p^{n-k}-1}})(H/K).$$

This map is an isomorphism that commutes with the appropriate restriction and transfer maps because the box product is the symmetric monoidal product in the category of Mackey functors.

Next we will define Φ_K for $K = C_{p^{k+1}}$. The subgroup H is now maximal in K, and

$$(N_{n-1}^n N_k^{n-1}\underline{M})(G/K) = (N_k^{n-1}\underline{M}_e \Box N_k^{n-1}\underline{M}_\gamma \Box \cdots \Box N_k^{n-1}\underline{M}_{\gamma^{p-1}})(C_{p^{n-1}}/K).$$

By Remark 1.2.2 it follows that

$$(N_k^{n-1}\underline{M}_e \,\Box\, N_k^{n-1}\underline{M}_\gamma \,\Box \cdots \Box\, N_k^{n-1}\underline{M}_{\gamma^{p-1}})(C_{p^{n-1}}/K) = \left(\left(\bigotimes_{i=0}^{p-1}(N_k^{n-1}\underline{M})(C_{p^{n-1}}/K)_{\gamma^i}\right) \oplus Im(tr_H^K)\right)/_{FR}$$

where $Im(tr_H^K) = \left(\underline{M}^{\Box|C_{p^{n-1}}/H|}\right)^{\Box|G/C_{p^{n-1}}|}(H/H)/_{W_K(H)}$. Further,

$$(N_k^{n-1}\underline{M})(C_{p^{n-1}}/K) \;=\; \left(\mathbb{Z}\{\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\} \oplus T\right)/_{FTR}.$$

The summand T is $(N_k^{n-1}\underline{M})(C_{p^{n-1}}/H)/_{W_K(H)}$ and letting β be $p^{n-k-2}-1$, we write $N(m_e \times m_{\gamma^p} \times \cdots \times m_{\gamma^{\beta p}})$ for a generator of $\mathbb{Z}\{\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\}$. When we quotient $(N_{n-1}^n N_k^{n-1}\underline{M})(G/K)$ out by the Frobenius reciprocity submodule we identify all elements in T with elements in $Im(tr_H^K)$, and hence,

$$(N_{n-1}^n N_k^{n-1}\underline{M})(G/K) \;\cong\; \left(\left(\bigotimes_{i=0}^{p-1}\mathbb{Z}\left\{\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\right\}_{\gamma^i}\right) \oplus Im(tr_H^K)\right)/_{\widetilde{FTR}}.$$

We denote a generator of the $(\gamma^i)^{th}$-copy of $\mathbb{Z}\{\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\}$ by either

$$N(m_{\gamma^i} \times m_{\gamma^{p+i}} \times \cdots \times m_{\gamma^{\beta p+i}})$$

or $N(\vec{m}_{\gamma^i})$ for short.

The elements that generate $\widetilde{FTR}$ result from fusing the FTR submodule of every $(N_k^{n-1}\underline{M})(G/K)_{\gamma^i}$ with Frobenius reciprocity. In particular, this fusion induces the following relations for all indices $\gamma^{\omega p+i}$, elements a and b in $\underline{M}(H/H)$, subgroups H'

in H and x in $\underline{M}(H/H')$.

$$\begin{aligned}
&N(\vec{m}_e)\otimes\cdots\otimes N(m_{\gamma^i}\times\cdots\times(a_{\gamma^{\omega p+i}}+b_{\gamma^{\omega p+i}})\times\cdots\times m_{\gamma^{\beta p+i}})\otimes\cdots\otimes N(\vec{m}_{\gamma^{p-1}})\\
&\qquad -N(\vec{m}_e)\otimes\cdots\otimes N(m_{\gamma^i}\times\cdots\times a_{\gamma^{\omega p+i}}\times\cdots\times m_{\gamma^{\beta p+i}})\otimes\cdots\otimes N(\vec{m}_{\gamma^{p-1}})\\
&\qquad -N(\vec{m}_e)\otimes\cdots\otimes N(m_{\gamma^i}\times\cdots\times b_{\gamma^{\omega p+i}}\times\cdots\times m_{\gamma^{\beta p+i}})\otimes\cdots\otimes N(\vec{m}_{\gamma^{p-1}})\\
&\approx\quad N(\vec{m}_e)\otimes\cdots\otimes tr_H^K(g(a_{\gamma^{\omega p+i}},b_{\gamma^{\omega p+i}}))\otimes\cdots\otimes N(\vec{m}_{\gamma^{p-1}})\\
&\approx\quad tr_H^K(res_H^K N(\vec{m}_e)\otimes\cdots\otimes g(a_{\gamma^{\omega p+i}},b_{\gamma^{\omega p+i}})\otimes\cdots\otimes res_H^K N(\vec{m}_{\gamma^{p-1}}))\\
&\approx\quad tr_H^K((m_e\otimes\cdots\otimes m_{\gamma^{\beta p}})^{\otimes p}\otimes\cdots\otimes g(a_{\gamma^{\omega p+i}},b_{\gamma^{\omega p+i}})\otimes\cdots\otimes(m_{\gamma^{p-1}}\otimes\cdots\otimes m_{\gamma^{p^{n-k-1}-1}})^{\otimes p})
\end{aligned}$$

and

$$\begin{aligned}
&N(\vec{m}_e)\otimes\cdots\otimes N(m_{\gamma^i}\times\cdots\times tr_{H'}^H(x)_{\gamma^{\omega p+i}}\times\cdots\times m_{\gamma^{\beta p+i}})\otimes\cdots\otimes N(\vec{m}_{\gamma^{p-1}})\\
&\approx\quad N(\vec{m}_e)\otimes\cdots\otimes tr_{H'}^K(f(x_{\gamma^{\omega p+i}}))\otimes\cdots\otimes N(\vec{m}_{\gamma^{p-1}})\\
&\approx\quad tr_H^K\left(res_H^K N(\vec{m}_e)\otimes\cdots\otimes tr_{H'}^H(f(x_{\gamma^{\omega p+i}}))\otimes\cdots\otimes res_H^K N(\vec{m}_{\gamma^{p-1}})\right)\\
&\approx\quad tr_H^K\left((\vec{m}_e)^{\otimes p}\otimes\cdots\otimes tr_{H'}^H(f(x_{\gamma^{\omega p+i}}))\otimes\cdots\otimes(\vec{m}_{\gamma^{p-1}})^{\otimes p}\right).
\end{aligned}$$

The polynomials $g(a_{\gamma^{\omega p+i}},b_{\gamma^{\omega p+i}})$ and $f(x_{\gamma^{\omega p+i}})$ are defined in Definition 2.2.5.

Further, because

$$\bigotimes_{i=0}^{p-1}\mathbb{Z}\left\{\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\right\}_{\gamma^i}$$

is isomorphic to

$$\mathbb{Z}\left\{\prod_{i=0}^{p-1}\left(\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\right)_{\gamma^i}\right\}$$

it follows that

$$\begin{aligned}
&(N_{n-1}^n N_k^{n-1}\underline{M})(G/K)\cong\\
&\qquad\left(\mathbb{Z}\left\{\prod_{i=0}^{p-1}\left(\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\right)_{\gamma^i}\right\}\oplus Im(tr_H^K)\right)/\widetilde{FTR}
\end{aligned}$$

and a generator of the first summand is

$$N(m_e \times m_{\gamma^p} \times \cdots \times m_{\gamma^{\beta p}} \times m_\gamma \times m_{\gamma^{p+1}} \times \cdots \times m_{\gamma^{\beta p+1}} \times \cdots$$
$$\cdots \times m_{\gamma^{p-1}} \times \cdots \times m_{\gamma^{\beta p+p-1}}),$$

which we abbreviate to

$$N\left(\vec{m}_e \times \vec{m}_\gamma \times \cdots \times \vec{m}_{\gamma^{p-1}}\right).$$

Then the submodule $\widetilde{FTR}$ is generated by elements of the form

$$\begin{aligned} & N\left(\vec{m}_e \times \cdots \times (m_{\gamma^i} \times \cdots \times \left(a_{\gamma^{\omega p+i}} + b_{\gamma^{\omega p+i}}\right) \times \cdots \times m_{\gamma^{\beta p+i}}\right) \times \cdots \times \vec{m}_{\gamma^{p-1}}\big) \\ - \quad & N\left(\vec{m}_e \times \cdots \times \left(m_{\gamma^i} \times \cdots \times a_{\gamma^{\omega p+i}} \times \cdots \times m_{\gamma^{\beta p+i}}\right) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\ - \quad & N\left(\vec{m}_e \times \cdots \times \left(m_{\gamma^i} \times \cdots \times b_{\gamma^{\omega p+i}} \times \cdots \times m_{\gamma^{\beta p+i}}\right) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\ - \quad & tr_H^K((\vec{m}_e)^{\otimes p} \otimes \cdots \otimes g(a_{\gamma^{\omega p+i}}, b_{\gamma^{\omega p+i}}) \otimes \cdots \otimes (\vec{m}_{\gamma^{p-1}})^{\otimes p}) \end{aligned}$$

and

$$\begin{aligned} & N\left(\vec{m}_e \times \cdots \times \left(m_{\gamma^i} \times \cdots \times tr_{H'}^H(x)_{\gamma^{\omega p+i}} \times \cdots \times m_{\gamma^{\beta p+i}}\right) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\ - \quad & tr_H^K((\vec{m}_e)^{\otimes p} \otimes \cdots \otimes tr_{H'}^H\left(f(x_{\gamma^{\omega p+i}})\right) \otimes \cdots \otimes (\vec{m}_{\gamma^{p-1}})^{\otimes p}) \end{aligned}$$

for all $e \leq \gamma^{\omega p+i} \leq \gamma^{\beta p+p-1}$, a and b in $\underline{M}(H/H)$, $H' < H$, and x in $\underline{M}(H/H')$. Since the Weyl group $W_G(K)$ acts by

$$\begin{aligned} & \gamma \cdot N\left(m_e \times \cdots \times m_{\gamma^{\beta p}} \times m_\gamma \times \cdots \times m_{\gamma^{\beta p+1}} \times \cdots \times m_{\gamma^{p-1}} \times \cdots \times m_{\gamma^{\beta p+p-1}}\right) = \\ & N\left(m_{\gamma^{\beta p+p-1}} \times m_{\gamma^{p-1}} \times \cdots \times m_{\gamma^{(\beta-1)p+p-1}} \times m_e \times \cdots \times m_{\gamma^{\beta p}} \times \cdots\right), \end{aligned}$$

the submodule $\widetilde{FTR}$ is Weyl equivariant.

Per contra,

$$(N_k^n \underline{M})(G/K) = \left(\mathbb{Z}\{\underline{M}(H/H)^{\times|G/K|}\} \oplus \underline{M}^{|G/H|}(H/H)/_{W_K(H)}\right)/_{FTR},$$

and a generator of the free summand is given by

$$N(m_e \times m_\gamma \times m_{\gamma^2} \times \cdots \times m_{\gamma^{p^{n-k-1}-1}}).$$

Therefore, we define the map

$$\Phi_K : (N_{n-1}^n N_k^{n-1} \underline{M})(G/K) \to (N_k^n \underline{M})(G/K)$$

by defining two maps

$$\begin{aligned} \phi_K &: \left(\underline{M}^{\Box|C_{p^{n-1}}/H|}\right)^{\Box|G/C_{p^{n-1}}|}(H/H)/_{W_K(H)} \to \underline{M}^{\Box|G/H|}(H/H)/_{W_K(H)} \\ \phi_K' &: \mathbb{Z}\left\{\prod_{i=0}^{p-1}\left(\underline{M}(H/H)^{\times|C_{p^{n-1}}/K|}\right)_{\gamma^i}\right\} \to \mathbb{Z}\left\{\underline{M}(H/H)^{\times|G/K|}\right\}. \end{aligned}$$

We define the map ϕ_K to be the isomorphism induced from the isomorphism

$$\Phi_H : \left(\underline{M}^{\Box|C_{p^{n-1}}/H|}\right)^{\Box|G/C_{p^{n-1}}|}(H/H) \to \underline{M}^{\Box|G/H|}(H/H)$$

described above, and the map ϕ_K' is the isomorphism that rearranges the indices of a generator $N(\vec{m}_e \times \vec{m}_\gamma \times \cdots \times \vec{m}_{\gamma^{p-1}})$. In particular, ϕ_K' maps

$$N\left(m_e \times m_{\gamma^p} \times \cdots \times m_{\gamma^{\beta p}} \times m_\gamma \times m_{\gamma^{p+1}} \times \cdots \times m_{\gamma^{\beta p+1}} \times \cdots \times m_{\gamma^{p-1}} \times \cdots \times m_{\gamma^{\beta p+p-1}}\right)$$

to

$$N(m_e \times m_\gamma \times m_{\gamma^2} \times \cdots \times m_{\gamma^{p^{n-k-1}-2}} \times m_{\gamma^{p^{n-k-1}-1}}).$$

The isomorphisms ϕ_K and $\phi_{K'}$ pass to a well-defined map Φ_K, and a Five Lemma argument analogous to the proof used in Theorem 2.2.4 shows that Φ_K is an isomorphism. Moreover, this map commutes with the appropriate restriction and transfer maps by construction.

Next we use induction to build isomorphisms

$$\Phi_{C_{p^i}} : (N^n_{n-1}N^{n-1}_k \underline{M})(G/C_{p^i}) \to (N^n_k \underline{M})(G/C_{p^i})$$

for $k+1 < i < n$. Using an argument similar to the one above we can show that $(N^n_{n-1}N^{n-1}_k \underline{M})(G/C_{p^i})$ is isomorphic to

$$\left(\mathbb{Z}\left\{ \prod_{r=0}^{p-1} \left(\underline{M}(H/H)^{\times |C_{p^{n-1}}/C_{p^i}|} \right)_{\gamma^r} \right\} \oplus (N^n_{n-1}N^{n-1}_k \underline{M})(G/C_{p^{i-1}})/_{W_{C_{p^i}}(C_{p^{i-1}})} \right) /_{\widetilde{FTR}}$$

where $\widetilde{FTR}$ is the submodule generated by

$$\begin{aligned}
& N\left(\vec{m}_e \times \cdots \times (m_{\gamma^r} \times \cdots \times (a_{\gamma^j} + b_{\gamma^j}) \times \cdots \times m_{\gamma^{\delta p + r}}) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\
- \quad & N\left(\vec{m}_e \times \cdots \times (m_{\gamma^r} \times \cdots \times a_{\gamma^j} \times \cdots \times m_{\gamma^{\delta p + r}}) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\
- \quad & N\left(\vec{m}_e \times \cdots \times (m_{\gamma^r} \times \cdots \times b_{\gamma^j} \times \cdots \times m_{\gamma^{\delta p + r}}) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\
- \quad & \sum_{H < K' < C_{p^i}} tr^{C_{p^i}}_{K'} \left(N\left((\vec{m}_e)^{\times |C_{p^i}/K'|} \times \cdots \times \sum_i \overrightarrow{(a_{\gamma^j} b_{\gamma^j})}^{K'}_i \times \cdots \times (\vec{m}_{\gamma^{p-1}})^{\times |C_{p^i}/K'|} \right)\right) \\
- \quad & tr^{C_{p^i}}_H \left((\vec{m}_e)^{\otimes |C_{p^i}/H|} \otimes \cdots \otimes g(a_{\gamma^j}, b_{\gamma^j}) \otimes \cdots \otimes (\vec{m}_{\gamma^{p-1}})^{\otimes |C_{p^i}/H|} \right)
\end{aligned}$$

and

$$\begin{aligned}
& N\left(\vec{m}_e \times \cdots \times (m_{\gamma^r} \times \cdots \times tr^H_{H'}(x)_{\gamma^j} \times \cdots \times m_{\gamma^{\delta p + r}}) \times \cdots \times \vec{m}_{\gamma^{p-1}}\right) \\
- \quad & tr^{C_{p^i}}_H \left((\vec{m}_e)^{\otimes p} \otimes \cdots \otimes tr^H_{H'}\left(f(x_{\gamma^j})\right) \otimes \cdots \otimes (\vec{m}_{\gamma^{p-1}})^{\otimes p} \right)
\end{aligned}$$

for $\delta = p^{n-i-1} - 1$ and for all $e \leq \gamma^j \leq \gamma^{\delta p + p - 1}$, a and b in $\underline{M}(H/H)$, $H' < H$, and x in $\underline{M}(H/H')$. The group

$$(N_k^n \underline{M})(G/C_{p^i}) = \left(\mathbb{Z}\left\{\underline{M}(H/H)^{\times|G/C_{p^i}|}\right\} \oplus (N_k^n \underline{M})(G/C_{p^i})/_{W_{C_{p^i}}(C_{p^{i-1}})}\right)/_{FTR}.$$

We can define a map $\Phi_{C_{p^i}}$ by defining two maps

$$\begin{aligned}
\phi_{C_{p^i}} &: (N_{n-1}^n N_k^{n-1} \underline{M})(G/C_{p^{i-1}})/_{W_{C_{p^i}}(C_{p^{i-1}})} \to (N_k^n \underline{M})(G/C_{p^i})/_{W_{C_{p^i}}(C_{p^{i-1}})} \\
\phi'_{C_{p^i}} &: \mathbb{Z}\left\{\prod_{i=0}^{p-1}\left(\underline{M}(H/H)^{\times|C_{p^{n-1}}/C_{p^i}|}\right)_{\gamma^i}\right\} \to \mathbb{Z}\left\{\underline{M}(H/H)^{\times|G/C_{p^i}|}\right\}.
\end{aligned}$$

We can assume by induction that

$$\Phi_{C_{p^{i-1}}} : (N_{n-1}^n N_k^{n-1} \underline{M})(G/C_{p^{i-1}}) \to (N_k^n \underline{M})(G/C_{p^{i-1}})$$

is an isomorphism, and so we define $\phi_{C_{p^i}}$ to be the corresponding induced isomorphism. The map $\phi'_{C_{p^i}}$ is the isomorphism that appropriately rearranges the indices of the generators of $\mathbb{Z}\left\{\prod_{i=0}^{p-1}\left(\underline{M}(H/H)^{\times|C_{p^{n-1}}/C_{p^i}|}\right)_{\gamma^i}\right\}$. These isomorphisms pass to a well-defined map $\Phi_{C_{p^i}}$, and, as above, a Five Lemma argument will show that $\Phi_{C_{p^i}}$ is an isomorphism. Moreover, it is clear by construction that this isomorphism commutes with the restriction and transfer maps.

Finally, it remains to define an isomorphism

$$\Phi_G : (N_{n-1}^n N_k^{n-1} \underline{M})(G/G) \to (N_k^n \underline{M})(G/G).$$

The group

$$\begin{aligned}
&(N_k^n \underline{M})(G/G) = \\
&\quad \left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus (N_k^n \underline{M})(G/C_{p^{n-1}})/_{W_G(C_{p^{n-1}})}\right)/_{FTR},
\end{aligned}$$

and $(N^n_{n-1}N^{n-1}_k\underline{M})(G/G)$ simplifies nicely. Indeed,

$$(N^n_{n-1}N^{n-1}_k\underline{M})(G/G) =$$
$$\Big(\mathbb{Z}\{(N^{n-1}_k\underline{M})(C_{p^{n-1}}/C_{p^{n-1}})\} \oplus \overbrace{(N^n_{n-1}N^{n-1}_k\underline{M})(G/C_{p^{n-1}})/_{W_G(C_{p^{n-1}})}}^{Im(tr^G_{C_{p^{n-1}}})}\Big)/_{TR},$$

and

$$(N^{n-1}_k\underline{M})(C_{p^{n-1}}/C_{p^{n-1}}) =$$
$$\Big(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \overbrace{(N^{n-1}_k\underline{M})(C_{p^{n-1}}/C_{p^{n-2}})/_{W_{C_{p^{n-1}}}(C_{p^{n-2}})}}^{T}\Big)/_{FTR}.$$

Thus, we will write

$$(N^n_{n-1}N^{n-1}_k\underline{M})(G/G) = \Big(\mathbb{Z}\{(\mathbb{Z}\{\underline{M}(H/H)\} \oplus T)\,/_{FTR}\} \oplus Im(tr^G_{C_{p^{n-1}}})\Big)\,/_{TR},$$

and we will let $n(a)$ denote a generator of the free summand $\mathbb{Z}\{\underline{M}(H/H)\}$ in $(N^{n-1}_k\underline{M})(C_{p^{n-1}}/C_{p^{n-1}})$. We can identify all elements of T with elements of $Im(tr^G_{C_{p^{n-1}}})$, and via Property 1.4.4 the formula for $N(n(a+b))$ must be isomorphic to the formula for $N(a+b)$ in $(N^n_k\underline{M})(G/G)$. Similarly, the formula for $N(n(tr^H_{H'}(x)))$ must be isomorphic to the formula for $N(tr^H_{H'}(x))$ in $(N^n_k\underline{M})(G/G)$. Hence, $(N^n_{n-1}N^{n-1}_k\underline{M})(G/G)$ is isomorphic to

$$\Big(\mathbb{Z}\{\underline{M}(H/H)\} \oplus (N^n_{n-1}N^{n-1}_k\underline{M})(G/C_{p^{n-1}})/_{W_G(C_{p^{n-1}})}\Big)\,/_{FTR},$$

and thus we can build an isomorphism Φ_G by defining the two isomorphisms

$$\phi_G : (N^n_{n-1}N^{n-1}_k\underline{M})(G/C_{p^{n-1}})/_{W_G(C_{p^{n-1}})} \to (N^n_k\underline{M})(G/G)/_{W_G(C_{p^{n-1}})}$$

and

$$\phi'_G : \mathbb{Z}\{\underline{M}(H/H)\} \to \mathbb{Z}\{\underline{M}(H/H)\}.$$

The map ϕ_G is induced from the isomorphism $\Phi_{C_{p^{n-1}}}$, and we let ϕ'_G be the identity. Therefore, we have defined an isomorphism $\Phi : N^n_{n-1}N^{n-1}_k\underline{M} \to N^n_k\underline{M}$. It follows that $N^G_K N^K_H \underline{M}$ is isomorphic to $N^G_H\underline{M}$ whenever $H < K < G$. $\square$

Corollary 2.2.8. *The map $N^G_H : \mathcal{M}ack_H \to \mathcal{M}ack_G$ is a strong symmetric monoidal functor for all subgroups H of G.*

Proof. Using the notation of the previous proof, we denote N^G_H by N^n_k. By Theorem 2.2.7, the map N^n_k is isomorphic to the composition

$$N^n_{n-1}N^{n-1}_{n-2}\cdots N^{k+2}_{k+1}N^{k+1}_k,$$

and by Theorems 2.2.3 and 2.2.4 each N^i_{i-1} is a strong symmetric monoidal functor. Therefore, the map N^G_H is a strong symmetric monoidal functor as well. $\square$

2.3 The New G-Symmetric Monoidal Structure

We can now use these norm functors $N^G_H : \mathcal{M}ack_H \to \mathcal{M}ack_G$ to endow the category of G-Mackey functors with a G-symmetric monoidal structure.

Theorem 2.3.1. *The functor $(-) \otimes (-) : \mathscr{S}et^{Fin,Iso}_G \times \mathcal{M}ack_G \to \mathcal{M}ack_G$ given by*

- $\emptyset \otimes \underline{M} := \underline{A}$

- $G/H \otimes \underline{M} := N_H^G i_H^* \underline{M}$ *for all orbits* G/H *of* G, *and*

- $(X \amalg Y) \otimes \underline{M} := (X \otimes \underline{M}) \square (Y \otimes \underline{M})$ *for all* X *and* Y *in* $\mathscr{S}et_G^{Fin}$

is a G*-symmetric monoidal structure on* $\mathcal{M}ack_G$.

Proof. The functor $(-) \otimes (-)$ is symmetric monoidal in the second factor because the functor N_H^G is strong symmetric monoidal. Moreover, if we restrict $(-) \otimes (-)$ down to $\mathscr{S}et^{Fin,Iso} \times \mathcal{M}ack_G$, then $X \otimes \underline{M} = (* \otimes \underline{M})^{\square |X|}$, and since

$$* \otimes \underline{M} = G/G \otimes \underline{M} = N_G^G i_G^* \underline{M} = \underline{M}$$

it follows that this restriction is the canonical exponentiation map.

It remains to show that $(-) \otimes (-)$ satisfies Property 3 of Definition 2.1.1. It will suffice to show that $(G/H \times G/K) \otimes \underline{M} = G/H \otimes (G/K \otimes \underline{M})$ for all orbits G/K and G/H. Assume that $H = C_{p^k}$, $K = C_{p^i}$, and $i < k$. Then the G-set $G/H \times G/K$ is isomorphic to $\amalg_{p^{n-k}} G/K$, and so

$$\begin{aligned} (G/H \times G/K) \otimes \underline{M} &\cong (\amalg_{p^{n-k}} G/K) \otimes \underline{M} \\ &\cong (G/K \otimes \underline{M})^{\square p^{n-k}} \\ &\cong (N_K^G i_K^* \underline{M})^{\square p^{n-k}} \\ &\cong N_K^G i_K^* (\underline{M}^{\square p^{n-k}}). \end{aligned}$$

Furthermore, $G/H \otimes (G/K \otimes \underline{M}) = N_H^G i_H^* N_K^G i_K^* \underline{M}$, but by Theorem 2.2.7

$$\begin{aligned} i_H^* N_K^G i_K^* \underline{M} &\cong i_H^* N_H^G N_K^H i_K^* \underline{M} \\ &= (N_K^H i_K^* \underline{M})^{\square p^{n-k}}. \end{aligned}$$

Because we are considering $(N_K^H i_K^* \underline{M})^{\Box p^{n-k}}$ as an H-Mackey functor there is no $W_G(H)$-action, and hence by Theorem 2.2.4,

$$(N_K^H i_K^* \underline{M})^{\Box p^{n-k}} \cong N_K^H i_K^* (\underline{M}^{\Box p^{n-k}}),$$

and it follows that

$$\begin{aligned} G/H \otimes G/K \otimes \underline{M} &\cong N_H^G N_K^H i_K^* (\underline{M}^{\Box p^{n-k}}) \\ &\cong N_K^G i_K^* (\underline{M}^{\Box p^{n-k}}). \end{aligned}$$

Therefore, $G/H \otimes (G/K \otimes \underline{M})$ is isomorphic to $(G/H \times G/K) \otimes \underline{M}$, and we have endowed the category of G-Mackey functors with a G-symmetric monoidal structure.

$\Box$

2.3.1 Tambara Functors are the G-Commutative Monoids

Finally, we need to show that Tambara functors are now the G-commutative monoids by proving the following theorem.

Theorem 2.3.2. *A G-Mackey functor $\underline{M}$ has the structure of a G-Tambara functor if and only if the functor $(-) \otimes \underline{M} : \mathscr{S}et_G^{Fin,Iso} \to \mathcal{M}ack_G$ extends to a functor $(-) \otimes \underline{M} : \mathscr{S}et_G^{Fin} \to \mathcal{M}ack_G$.*

$$\begin{tikzcd} \mathscr{S}et_G^{Fin,Iso} \arrow[r, "(-)\otimes\underline{M}"] \arrow[d] & \mathcal{M}ack_G \\ \mathscr{S}et_G^{Fin} \arrow[ur, dashed] & \end{tikzcd}$$

Before proving this theorem we need to extend the norm functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ to a functor $\mathcal{N}_H^G : Tamb_H \to Tamb_G$ of Tambara functors, and show that this extension is left adjoint to the restriction functor $i_H^* : Tamb_G \to Tamb_H$ that takes a G-Tambara functor to its underlying H-Tambara functor.

Lemma 2.3.3. *For all subgroups H of G the functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ canonically extends to a functor $\mathcal{N}_H^G : Tamb_H \to Tamb_G$.*

Proof. By Theorem 2.2.7 it suffices to let H be the maximal subgroup of G. To show that the norm functor N_H^G extends to a functor $\mathcal{N}_H^G$ of Tambara functors we need to show that if $\underline{S}$ is an H-Tambara functor then $N_H^G\underline{S}$ is a G-Tambara functor. To begin, let $Green_H$ and $Green_G$ be the categories of H-Green functors and G-Green functors, respectively. Since $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ is strong symmetric monoidal it extends to a functor $N_H^G : Green_H \to Green_G$ such that the diagram below commutes where the vertical maps are the forgetful functors that send a Green functor to its underlying Mackey functor.

$$\begin{tikzcd} Green_H \arrow[r, "N_H^G"] \arrow[d] & Green_G \arrow[d] \\ \mathcal{M}ack_H \arrow[r, "N_H^G"] & \mathcal{M}ack_G \end{tikzcd}$$

Thus, given an H-Tambara functor $\underline{S}$ the G-Mackey functor $N_H^G\underline{S}$ will be a G-Tambara functor if we can define the internal norm maps $N_{K'}^K : (N_H^G\underline{S})(G/K') \to (N_H^G\underline{S})(G/K)$ for all subgroups K and K' of G. If K and K' are subgroups of H such that $K' < K$ then we denote the corresponding internal norm map of $\underline{S}$ by

$\tilde{N}_{K'}^{K} : \underline{S}(H/K') \to \underline{S}(H/K)$. Moreover, in ([12], Proposition 9.1) Strickland shows that the box product is the coproduct in the category of Tambara functors. Hence, we can define the internal norm

$$N_{K'}^{K} : \underline{S}^{\Box|G/H|}(H/K') \to \underline{S}^{\Box|G/H|}(H/K)$$

of $N_H^G\underline{S}$ by defining $N_{K'}^{K}$ to be $(\tilde{N}_{K'}^{K})^{\Box|G/H|}$.

It remains to define the internal norm maps $N_K^G : (N_H^G\underline{S})(G/K) \to (N_H^G\underline{S})(G/G)$ for all subgroups K in G. First, define the internal norm map $N_H^G : (N_H^G\underline{S})(G/H) \to (N_H^G\underline{S})(G/G)$ by the composition

$$\underline{S}^{\Box|G/H|}(H/H) \xrightarrow{m} \underline{S}(H/H) \xrightarrow{N} \left(\mathbb{Z}\{\underline{S}(H/H)\} \oplus Im(tr_H^G)\right)/_{TR}.$$

The map m is the multiplication map of Definition 1.3.1, the map N is given in Definition 2.2.2. If K is a subgroup of H then we let the internal norm map N_K^G : $(N_H^G\underline{S})(G/K) \to (N_H^G\underline{S})(G/G)$ be the composition

$$\underline{S}^{\Box|G/H|}(H/K) \xrightarrow{m} \underline{S}(H/K) \xrightarrow{\tilde{N}_K^H} \underline{S}(H/H) \xrightarrow{N} (N_H^G\underline{S})(G/G).$$

The nature of the construction of $(N_H^G\underline{S})(G/G)$ forces these compositions to support all of the properties of an internal norm map described in Definition 1.4.2. It follows that the functor $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ extends to a functor $\mathcal{N}_H^G : Tamb_H \to Tamb_G$. □

Lemma 2.3.4. *For all subgroups H in G the functor $\mathcal{N}_H^G : Tamb_H \to Tamb_G$ is left adjoint to the restriction functor $i_H^* : Tamb_G \to Tamb_H$.*

Proof. Let H be the maximal subgroup of G and $Tamb_G(\underline{S}, \underline{S}')$ be the set of morphisms $\underline{S} \to \underline{S}'$. By Theorem 2.2.7 and the following formal property of adjunctions it will suffice to show that there is a natural bijection

$$F : Tamb_H(\underline{S}, i_H^*\underline{R}) \to Tamb_G(\mathcal{N}_H^G\underline{S}, \underline{R})$$

for all $\underline{R}$ in $Tamb_G$ and $\underline{S}$ in $Tamb_H$.

Property of Adjunctions 2.3.1. Suppose that $L : \mathcal{C} \to \mathcal{C}'$ and $R : \mathcal{C}' \to \mathcal{C}$ form an adjoint pair, as do the functors $L' : \mathcal{C}' \to \mathcal{D}$ and $R' : \mathcal{D} \to \mathcal{C}'$. Then the composite functors $L'L : \mathcal{C} \to \mathcal{D}$ and $RR' : \mathcal{D} \to \mathcal{C}$ yield an adjoint pair as well [9].

A morphism Ψ in $Tamb_G(\mathcal{N}_H^G\underline{S}, \underline{R})$ consists of a collection of ring homomorphisms

$$\{\Psi_K : (\mathcal{N}_H^G\underline{S})(G/K) \to \underline{R}(G/K) : K \leq G\}$$

that commute with the appropriate restriction, transfer and norm maps. But, we have defined $(\mathcal{N}_H^G\underline{S})(G/G)$ by

$$(\mathcal{N}_H^G\underline{S})(G/G) = \left(\mathbb{Z}\{\underline{S}(H/H)\} \oplus Im(tr_H^G)\right)/_{TR},$$

and every element in $(\mathcal{N}_H^G\underline{S})(G/G)$ is either in the image of the transfer map tr_H^G or is a sum of elements in the image of the norm map N_H^G. (Indeed, every generator $N(a)$ in $\mathbb{Z}\{\underline{S}(H/H)\}$ is the norm of the element $a \otimes 1^{\otimes |G/H|-1}$ in $(\mathcal{N}_H^G\underline{S})(G/H)$.) Thus, since we require $N_H^G\Psi_H$ to equal $\Psi_G N_H^G$ and $tr_H^G\Psi_H$ to equal $\Psi_G tr_H^G$, the ring homomorphism $\Psi_G : (\mathcal{N}_H^G\underline{S})(G/G) \to \underline{R}(G/G)$ is completely determined by Ψ_H. Moreover, for all

subgroups K of H

$$(\mathcal{N}_H^G \underline{S})(G/K) = \underline{M}^{\Box|G/H|}(H/K),$$

and hence a morphism $\Psi : N_H^G \underline{S} \to \underline{R}$ is completed determined by a collection of ring homomorphisms

$$\{\Psi_K : \underline{M}^{\Box|G/H|}(H/K) \to \underline{R}(G/K) : K \leq H\}.$$

By Remark 1.2.3 the above set of maps determines and is determined by a collection of $W_G(K)$-equivariant maps

$$\{\theta_K : \underline{S}(H/K)^{\otimes|G/H|} \to \underline{R}(G/K) : K \leq H\}$$

such that whenever K is a subgroup of K' the maps θ_K and $\theta_{K'}$ must satisfy

$$\begin{aligned}
\theta_{K'} \circ id^{\otimes i-1} \otimes tr_K^{K'} \otimes id^{\otimes p-i} &= tr_K^{K'} \circ \theta_K \circ (res_K^{K'})^{\otimes i-1} \otimes id \otimes (res_K^{K'})^{\otimes p-i} \text{ for all } i \\
\theta_K \circ (res_K^{K'})^{\otimes p} &= res_K^{K'} \circ \theta_{K'} \qquad (2.3.1) \\
\theta_{K'} \circ (N_K^{K'})^{\otimes p} &= N_K^{K'} \circ \theta_K.
\end{aligned}$$

But, each θ_K is determined by a $W_H(K)$-equivariant map $\Phi_K : \underline{S}(H/K) \to \underline{R}(G/K)$ that commutes with the appropriate restriction, transfer and norm maps because we can define

$$\begin{aligned}
\theta_K(s_e \otimes s_\gamma \otimes \cdots \otimes s_{\gamma^{p-1}}) &= \theta_K((s_e \otimes 1^{\otimes p-1})\gamma \cdot (s_\gamma \otimes 1^{\otimes p-1}) \cdots \gamma^{p-1} \cdot (s_{\gamma^{p-1}} \otimes 1^{\otimes p-1})) \\
&= \Phi_K(s_e)\gamma \cdot \Phi_K(s_\gamma) \cdots \gamma^{p-1} \cdot \Phi_K(s_{\gamma^{p-1}}).
\end{aligned}$$

It is straightforward to verify that Relations 2.3.1 hold. For example,

$$
\begin{aligned}
&(\theta_{K'} \circ id^{\otimes p-1} \otimes tr_K^{K'})(s_e \otimes s_\gamma \otimes \cdots \otimes s_{\gamma^{p-1}}) \\
&= \theta_{K'}(s_e \otimes s_\gamma \otimes \cdots \otimes tr_K^{K'}(s_{\gamma^{p-1}})) \\
&= \Phi_{K'}(s_e)\gamma \cdot \Phi_{K'}(s_\gamma) \cdots \gamma^{p-1} \cdot \Phi_{K'}(tr_K^{K'}(s_{\gamma^{p-1}})) \\
&= \Phi_{K'}(s_e)\gamma \cdot \Phi_{K'}(s_\gamma) \cdots tr_K^{K'}(\gamma^{p-1} \cdot \Phi_K(s_{\gamma^{p-1}})) \\
&= tr_K^{K'}(res_K^{K'}\Phi_{K'}(s_e)\gamma \cdot res_K^{K'}\Phi_{K'}(s_\gamma) \ldots \gamma^{p-1} \cdot \Phi_K(s_{\gamma^{p-1}})) \text{ (Frobenius reciprocity)} \\
&= tr_K^{K'}(\Phi_K res_K^{K'}(s_e)\gamma \cdot \Phi_K res_K^{K'}(s_\gamma) \cdots \gamma^{p-1} \cdot \Phi_K(s_{\gamma^{p-1}})) \\
&= (tr_K^{K'} \circ \theta_K \circ (res_K^{K'})^{\otimes p-1} \otimes id)(s_e \otimes s_\gamma \otimes \cdots \otimes s_{\gamma^{p-1}}).
\end{aligned}
$$

In fact, the maps Φ_K are in one-to-one correspondence with the maps θ_K. If $\theta_K = \theta'_K$ then $\theta_K(s_e \otimes 1 \otimes \cdots \otimes 1) = \theta'_K(s_e \otimes 1 \otimes \cdots \otimes 1)$. Hence, $\Phi_K(s_e) = \Phi'_K(s_e)$ for all s_e in $\underline{S}(H/K)$, making $\Phi_K = \Phi'_K$.

Therefore, given Φ in $Tamb_H(\underline{S}, i_H^*\underline{R})$, we define the bijection

$$F : Tamb_H(\underline{S}, i_H^*\underline{R}) \to Tamb_G(\mathcal{N}_H^G\underline{S}, \underline{R})$$

by letting $F(\Phi) : \mathcal{N}_H^G\underline{S} \to \underline{R}$ be the morphism determined by the collection of ring homomorphisms

$$\{F(\Phi)_K : \underline{S}(H/K)^{\otimes|G/H|} \to (i_H^*\underline{R})(H/K) : K \leq H\}$$

where each $F(\Phi)_K$ is given by

$$F(\Phi)_K(s_e \otimes s_\gamma \otimes \cdots \otimes s_{\gamma^{p-1}}) = \Phi_K(s_e)\gamma \cdot \Phi_K(s_\gamma) \cdots \gamma^{p-1} \cdot \Phi_K(s_{\gamma^{p-1}}).$$

$\square$

We now prove Theorem 2.3.2.

Proof. Let $\underline{S}$ be a G-Tambara functor.Given any map $G/K \to G/H$ between orbits we need to define an induced map $G/K \otimes \underline{S} \to G/H \otimes \underline{S}$ of Mackey functors. By Lemma 2.3.4 there is a counit map $\mathcal{N}_K^H i_K^* \underline{S} \to i_H^* \underline{S}$ from the adjunction between $\mathcal{N}_K^H$ and i_K^*. We obtain the map induced from $G/K \to G/H$ by applying $\mathcal{N}_H^G$ to the above counit map. The result is a map $\mathcal{N}_H^G \mathcal{N}_K^H i_K^* \underline{S} \to N_H^G i_H^* \underline{S}$, which by Theorem 2.2.7 is a map $G/K \otimes \underline{S} \to G/H \otimes \underline{S}$.

Now assume that $(-) \otimes \underline{M} : \mathscr{S}et_G^{Fin,Iso} \to \mathcal{M}ack_G$ extends to a functor $\mathscr{S}et_G^{Fin} \to \mathcal{M}ack_G$, and so every map $X \to Y$ of finite G-sets induces a map of Mackey functors $X \otimes \underline{M} \to Y \otimes \underline{M}$. We will show that this property endows $\underline{M}$ with the structure of a G-Tambara functor. First, the Mackey functor $\underline{M}$ will be a commutative Green functor if we can define a multiplication map $m : \underline{M} \Box \underline{M} \to \underline{M}$ and a unit map $1_{\underline{M}} : \underline{A} \to \underline{M}$ such that the diagrams in Definition 1.3.1 commute.

Letting $*$ be the orbit G/G, so that $* \otimes \underline{M} = \underline{M}$, we define the multiplication map m to be the map induced by the projection map $p : * \amalg * \to *$. Moreover, the inclusion map $i : \emptyset \hookrightarrow *$ induces the unit map, and if we apply the functor $(-) \otimes \underline{M}$ to the following three diagrams in $\mathscr{S}et_G^{Fin}$ then we obtain the desired commutative diagrams in $\mathcal{M}ack_G$.

$$\begin{tikzcd}
* \amalg * \amalg * \arrow[r, "id \amalg p"] \arrow[d, "p \amalg id"'] & * \amalg * \arrow[d, "p"] \\
* \amalg * \arrow[r, "p"] & *
\end{tikzcd}$$

$$\begin{tikzcd} \emptyset \amalg * \arrow[r, "i\amalg id"] \arrow[dr, "="'] & * \amalg * & \emptyset \amalg * \arrow[l, "i\amalg id"'] \arrow[dl, "="] \\ & * & \end{tikzcd} \qquad \begin{tikzcd} * \amalg * \arrow[rr, "\tau"] \arrow[dr, "p"'] & & * \amalg * \arrow[dl, "p"] \\ & * & \end{tikzcd}$$

The commutative Green functor $\underline{M}$ will be a Tambara functor if it supports internal norm maps $N_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ for all subgroups H and K of G. The map $H/K \to H/H$ of H-orbits induces a map $\nu_K^H : N_K^H i_K^* \underline{M} \to i_H^* \underline{M}$ of commutative Green functors, and regarding $(i_K^*\underline{M})(K/K)$ as $\underline{M}(G/K)$ we can define the internal norm map $N_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ to be the composition

$$\underline{M}(G/K) \xrightarrow{N} (N_K^H i_K^* \underline{M})(H/H) \xrightarrow{(\nu_K^H)_H} \underline{M}(G/H).$$

We defined the map $N : \underline{M}(G/K) \to (N_K^H i_K^* \underline{M})(H/H)$ in Definition 2.2.2. Furthermore, the construction of the norm functor $N_K^H : \mathcal{M}ack_K \to \mathcal{M}ack_H$ and the definition of the map N force this composition to maintain the necessary compatibility requirements described in Properties 4 and 5 of Definition 1.4.2. Since the functor $N_K^G : \mathcal{M}ack_K \to \mathcal{M}ack_H$ is isomorphic to the composition of functors $N_H^G N_K^H$ for all H, K, and G, it follows that the internal norm $N_K^G : \underline{M}(G/K) \to \underline{M}(G/G)$ equals the composition $N_H^G N_K^H$ of internal norms.

Lastly, we need to verify that $res_H^G N_H^G(m) = \prod_{\gamma \in W_G(H)} \gamma \cdot m$ for all m in $\underline{M}(G/H)$ and all subgroups H of G. The composition

$$\begin{aligned} res_H^G N_H^G(m) &= res_H^G (\nu_H^G)_G N(m) \\ &= (\nu_H^G)_H res_H^G N(m) \\ &= (\nu_H^G)_H (m^{\otimes |G/H|}) \end{aligned}$$

where the map $(\nu_H^G)_H : \underline{M}^{\square|G/H|}(G/H) \to \underline{M}(G/H)$ is induced from the map $i_H^*(G/H \to G/G)$. This map must be the fold map $\coprod_{\gamma \in W_G(H)} H/H \to H/H$ twisted by the Weyl action because the composition $G/H \xrightarrow{\gamma \cdot} G/H \to G/G$ agrees with the map $G/H \to G/G$. Therefore, $(\nu_H^G)_H$ is given by

$$m_1 \otimes m_2 \otimes \cdots \otimes m_{|G/H|} \mapsto m_1(\gamma \cdot m_2) \cdots (\gamma^{|G/H|-1} \cdot m_{|G/H|}),$$

and $res_H^G N_H^G(m) = \prod_{\gamma \in W_G(H)} \gamma \cdot m$. We conclude that $\underline{M}$ maintains the structure of a Tambara functor, and moreover that Tambara functors are the G-commutative monoids. $\square$

2.4 Two Interesting Consequences

2.4.1 Building Tambara Functors From Commutative Rings

There is an extra perk to the fact that the norm functor $\mathcal{N}_H^G : Tamb_H \to Tamb_G$ is left adjoint to the restriction functor $i_H^* : Tamb_G \to Tamb_H$. If H is the trivial subgroup and $\underline{S}$ is a G-Tambara functor then $i_H^* \underline{S}$ is simply the commutative ring $\underline{S}(G/e)$. Thus, we can use $\mathcal{N}_e^G$ to build a Tambara functor from any commutative ring.

Example 2.4.1. *Constructing the Tambara Functor* $\mathcal{N}_e^{C_2}(\mathbb{Z}/3)$.

The ring

$$\mathcal{N}_e^{C_2}(\mathbb{Z}/3)(C_2/e) = \mathbb{Z}/3 \otimes \mathbb{Z}/3 = \mathbb{Z}/3$$

with trivial C_2-action, and

$$\mathcal{N}_e^{C_2}(\mathbb{Z}/3)(C_2/C_2) = (\mathbb{Z}\{N(0), N(1), N(2)\} \oplus \mathbb{Z}/3)/_{TR}.$$

Then Tambara reciprocity induces the following relations.

$$\begin{aligned} N(0) &= 0 \qquad (2.4.1)\\ N(2) &= N(1+1) = N(1) + N(1) + tr_e^{C_2}(1 \otimes 1)\\ 0 &= N(1+2) = 3N(1) + tr_e^{C_2}(1 \otimes 1) + tr_e^{C_2}(1 \otimes 2) \end{aligned}$$

Hence, we can eliminate $N(0)$ and $N(2)$ as generators, $3N(1) = 0$, and as a group

$$\mathcal{N}_e^{C_2}(\mathbb{Z}/3)(C_2/C_2) = \mathbb{Z}/3\{N(1)\} \oplus \mathbb{Z}/3\{tr_e^{C_2}(1)\}.$$

We use Frobenius reciprocity to determine the ring structure. In particular,

$$tr_e^{C_2}(1)tr_e^{C_2}(1) = tr_e^{C_2}(1res_e^{C_2}(1)) = tr_e^{C_2}(2) = 2tr_e^{C_2}(1),$$

and so if we let $tr_e^{C_2}(1) = t$ then $\mathcal{N}_e^{C_2}(C_2/C_2) = \mathbb{Z}/3[t]/_{t^2=2t}$. Via Property 3 of Definition 2.2.1 we have $res_e^{C_2}(1) = 1$ and

$$res_e^{C_2}(t) = \sum_{\gamma \in C_2} \gamma \cdot 1 = 2.$$

It remains to determine the internal norm map $N_e^{C_2} : \mathbb{Z}/3 \to \mathbb{Z}/3[t]/_{t^2=2t}$, and as described in the proof of Lemma 2.3.3, we define this map via the composition

$$\mathbb{Z}/3 \otimes \mathbb{Z}/3 \xrightarrow{m} \mathbb{Z}/3 \xrightarrow{N} \mathbb{Z}/3[t]/_{t^2=2t}$$

where m is the multiplication map of $\mathbb{Z}/3$. The map $N : \mathbb{Z}/3 \to \mathcal{N}_e^{C_2}(\mathbb{Z}/3)(C_2/C_2)$ is the map of Definition 2.2.2, and thus if we combine this map with Relations 2.4.1 then

$$\begin{aligned} N(0) &= 0, \\ N(1) &= 1, \text{ and} \\ N(2) &= 2+t. \end{aligned}$$

Therefore, we define the internal norm map of $\mathcal{N}_e^{C_2}(\mathbb{Z}/3)$ by $N_e^{C_2}(1) = 1$ and $N_e^{C_2}(2) = 2+t$. We can now construct the Tambara functor diagram for $\mathcal{N}_e^{C_2}(\mathbb{Z}/3)$.

$$\begin{tikzcd}
1 \arrow[d, mapsto] & t \arrow[d, mapsto] & \mathbb{Z}/3[t]/_{t^2=2t} \arrow[d, bend right=50, "res_e^{C_2}"'] & t \\
1 & 2 & \mathbb{Z}/3 \arrow[u, "N_e^{C_2}"] \arrow[u, bend right=50, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto]
\end{tikzcd}$$

Example 2.4.2. *Constructing the Tambara Functor* $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)$.

Since the group $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/e)$ consists of $\mathbb{Z}[x]/x^2 \otimes \mathbb{Z}[x]/x^2$ with a Weyl action that swaps the tensor factors, as a ring,

$$\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/e) = \mathbb{Z}[x,y]/(x^2,y^2),$$

and if γ is the generator of C_2 then the C_2-action is given by $\gamma \cdot x = y$. The group $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2)$ is

$$\left(\mathbb{Z}\{\mathbb{Z}[x]/x^2\} \oplus (\mathbb{Z}[x,y]/(x^2,y^2))/_{C_2}\right)/_{TR},$$

but this simplifies nicely. When we quotient $\mathbb{Z}[x,y]/(x^2,y^2)$ out by the C_2-action we identify x with y, and so we will write the transfer summand as $\mathbb{Z}\{t,v,w\}$ where $t = tr_e^{C_2}(1)$, $v = tr_e^{C_2}(x)$, and $w = tr_e^{C_2}(xy)$. Further, by Tambara reciprocity we can write every generator $N(a+bx)$ of $\mathbb{Z}\{\mathbb{Z}[x]/x^2\}$ as a linear combination of $N(1)$, $N(x)$ and transfer terms. Hence, $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2)$ becomes

$$\mathbb{Z}[x]/x^2 \oplus \mathbb{Z}\{t,v,w\}.$$

We determine the multiplication using Frobenius reciprocity. For example,

$$xt = xtr_e^{C_2}(1) = tr_e^{C_2}(res_e^{C_2}(x)1) = tr_e^{C_2}(xy) = w, \text{ and}$$

$$vw = vtr_e^{C_2}(xy) = tr_e^{C_2}(res_e^{C_2}(v)xy) = tr_e^{C_2}((x+y)xy) = 0.$$

Thus, as a ring $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2)$ is isomorphic to

$$\mathbb{Z}[x,t,v,w]/_{xt=w,t^2=2t,tv=2v,tw=2w,v^2=w,x^2=xw=xv=vw=w^2=0}.$$

The internal norm map of $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)$ to is given by the composition

$$\mathbb{Z}[x]/x^2 \otimes \mathbb{Z}[x]/x^2 \xrightarrow{m} \mathbb{Z}[x]/x^2 \xrightarrow{N} \mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2),$$

and hence $N_e^{C_2}(x) = N_e^{C_2}(y) = x$. We show the Tambara functor diagram for $\mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)$ below. We used Property 3 of Definition 2.2.1 to determine the re-

striction map.

$$
\begin{tikzcd}
1 \arrow[d, mapsto] & t \arrow[d, mapsto] & v \arrow[d, mapsto] & w \arrow[d, mapsto] & x \arrow[d, mapsto] & \mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[d, bend right, "res_e^{C_2}"'] & t & v & v & w \\
1 & 2 & x+y & 2xy & xy & \mathbb{Z}[x,y]/(x^2,y^2) \arrow[u, "N_e^{C_2}"] \arrow[u, bend right, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto] & x \arrow[u, mapsto] & y \arrow[u, mapsto] & xy \arrow[u, mapsto]
\end{tikzcd}
$$

2.4.2 Green Functors Can Have Multiple Tambara Functor Structures

We can also use the norm functors $N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$ to endow a commutative Green functor with more than one Tambara functor structure.

Corollary 2.4.1. *If a Mackey functor $\underline{M}$ is a G-commutative monoid with endomorphisms that are not automorphisms then $\underline{M}$ will exhibit more than one distinct Tambara functor structure.*

Proof. Let $\underline{M}$ in $\mathcal{M}ack_G$ be a G-commutative monoid. Then by Theorem 2.3.2 the Mackey functor $\underline{M}$ is Tambara functor, and as described in the proof of Theorem 2.3.2 for all subgroups K and H of G we can define the internal norm maps $N_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ of $\underline{M}$ by the composition

$$\underline{M}(G/K) \xrightarrow{N} (N_K^H i_K^* \underline{M})(H/H) \xrightarrow{(\nu_H^G)_H} \underline{M}(G/H).$$

However, if a map $\phi : \underline{M} \to \underline{M}$ is an endomorphism but not an automorphism then

the maps $\widetilde{N}_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ given by the composition

$$\underline{M}(G/K) \xrightarrow{N} (N_K^H i_K^* \underline{M})(H/H) \xrightarrow{(\nu_H^G)_H} \underline{M}(G/H) \xrightarrow{\phi_H} \underline{M}(G/H)$$

also define valid internal norm maps. Hence, we have constructed another distinct Tambara functor structure on $\underline{M}$. $\square$

Example 2.4.3. We will endow the C_2-Tambara functor $F_T(\mathbb{Z}[x]/x^2)$ defined in Example 1.4.8 with infinitely many Tambara functor structures. The Tambara functor $F_T(\mathbb{Z}[x]/x^2)$ is given by the diagram below.

$$\begin{tikzcd}
1 \arrow[d, mapsto] & t \arrow[d, mapsto] & u \arrow[d, mapsto] & n \arrow[d, mapsto] & F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[d, bend right=60, "res_e^{C_2}"'] & t & u \\
1 & 2 & 2x & 0 & \mathbb{Z}[x]/x^2 \arrow[u, "N_e^{C_2}"] \arrow[u, bend right=60, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto] & x \arrow[u, mapsto]
\end{tikzcd}$$

The ring $F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) = \mathbb{Z}[t,u,n]/_{(t^2=2t, tu=2u, tn=nu=u^2=n^2=0)}$, and $N_e^{C_2}(x) = n$.

We can forget the Tambara functor structure of $F_T(\mathbb{Z}[x]/x^2)$ by eliminating the norm map and consider $F_T(\mathbb{Z}[x]/x^2)$ as a commutative Green functor. We can then reconstruct its Tambara functor structure by building $N_e^{C_2} i_e^*(F_T(\mathbb{Z}[x]/x^2))$ and a commutative Green functor map $\nu_e^{C_2} : N_e^{C_2} i_e^*(F_T(\mathbb{Z}[x]/x^2)) \to F_T(\mathbb{Z}[x]/x^2)$. But, since $i_e^*(F_T(\mathbb{Z}[x]/x^2)) = \mathbb{Z}[x]/x^2$, we have already accomplished much of the heavy lifting in Example 2.4.2. In Example 2.4.2 we discovered that

$$N_e^{C_2} i_e^*(F_T(\mathbb{Z}[x]/x^2))(C_2/C_2) \cong$$

$$\mathbb{Z}[x,t,v,w]/_{xt=w, t^2=2t, tv=2v, tw=2w, v^2=w, x^2=xw=xv=vw=w^2=0},$$

and thus the commutative Green functor $N_e^{C_2} i_e^*(F_T(\mathbb{Z}[x]/x^2))$ is

$$
\begin{tikzcd}
1 \arrow[d, mapsto] & t \arrow[d, mapsto] & z \arrow[d, mapsto] & w \arrow[d, mapsto] & x \arrow[d, mapsto] & \mathcal{N}_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[d, bend right, "res_e^{C_2}"'] & t & v & v & w \\
1 & 2 & x+y & 2xy & xy & \mathbb{Z}[x,y]/(x^2,y^2) \arrow[u, bend right, "tr_e^{C_2}"'] & 1 \arrow[u, mapsto] & x \arrow[u, mapsto] & y \arrow[u, mapsto] & xy \arrow[u, mapsto]
\end{tikzcd}
$$

We can define the map $\nu_e^{C_2} : N_e^{C_2} i_e^*(\mathbb{Z}[x]/x^2) \to F_T(\mathbb{Z}[x]/x^2)$ by letting

$$(\nu_e^{C_2})_e : \mathbb{Z}[x,y]/(x^2,y^2) \to \mathbb{Z}[x]/x^2$$

be the map induced from the multiplication map

$$\mathbb{Z}[x]/x^2 \otimes \mathbb{Z}[x]/x^2 \to \mathbb{Z}[x]/x^2$$

and $(\nu_e^{C_2})_{C_2}(x)$ equal n. Because we need $(\nu_e^{C_2})_{C_2} \circ tr$ to equal $tr \circ (\nu_e^{C_2})_e$ we define

$$
\begin{aligned}
(\nu_e^{C_2})_{C_2}(t) &= t, \\
(\nu_e^{C_2})_{C_2}(v) &= u, \text{ and} \\
(\nu_e^{C_2})_{C_2}(w) &= 0
\end{aligned}
$$

The internal norm $N_e^{C_2}$ of $F_T(\mathbb{Z}[x]/x^2)$ is given by the composition

$$\mathbb{Z}[x]/x^2 \xrightarrow{N} (N_e^{C_2})(\mathbb{Z}[x]/x^2)(C_2/C_2) \xrightarrow{(\nu_e^{C_2})_{C_2}} F_T(\mathbb{Z}[x]/x^2)(C_2/C_2).$$

Hence, $N_e^{C_2}(x) = n$, and we recover the Tambara functor structure discussed in Example 1.4.8.

However, given integers k and s, we can define an endomorphism $\psi^{k,s}$ of $F_T(\mathbb{Z}[x]/x^2)$

$$\begin{tikzcd}
F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[r, "\psi^{k,s}_{C_2}"] \arrow[d, bend right] & F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[d, bend right] \\
\mathbb{Z}[x]/x^2 \arrow[r, "\psi^{k,s}_{e}"] \arrow[u, bend right] & \mathbb{Z}[x]/x^2 \arrow[u, bend right]
\end{tikzcd}$$

by defining $\psi^{k,s}_e(x) = kx$ and $\psi^{k,s}_{C_2}(t) = t$, $\psi^{k,s}_{C_2}(u) = ku$, and $\psi^{k,s}_{C_2}(n) = sn$.

We can then define another internal norm map of $F_T(\mathbb{Z}[x]/x^2)$ by the composition

$$\begin{tikzcd}
\mathbb{Z}[x]/x^2 \arrow[r, "N"] & N_e^{C_2}(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[r, "(\nu_e^{C_2})_{C_2}"] & F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) \arrow[r, "\psi^{k,s}_{C_2}"] & F_T(\mathbb{Z}[x]/x^2)(C_2/C_2) \\
x \arrow[r, mapsto] & N(x) \arrow[r, mapsto] & n \arrow[r, mapsto] & sn
\end{tikzcd}$$

There are infinitely many endomorphisms $\psi^{k,s}$, and they give rise to infinitely many internal norm maps $N_e^{C_2} : \mathbb{Z}[x]/x^2 \to F_T(\mathbb{Z}[x]/x^2)(C_2/C_2)$. Therefore, we can define infinitely many distinct Tambara functor structures on $F_T(\mathbb{Z}[x]/x^2)$.

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